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The Q -operator and functional relations of the eight-vertex model at root-of-unity $\eta = \frac{2mK}{N}$ for odd N

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Abstract

Following Baxter's method of producing Q_{72} -operator, we construct the Q -operator of the root-of-unity eight-vertex model for the crossing parameter $\eta = \frac{2mK}{N}$ with odd N where Q_{72} does not exist. We use this new Q -operator to study the functional relations in the Fabricius–McCoy comparison between the root-of-unity eight-vertex model and the superintegrable N -state chiral Potts model. By the compatibility of the constructed Q -operator with the structure of Baxter's eight-vertex (solid-on-solid) SOS model, we verify the set of functional relations of the root-of-unity eight-vertex model using the explicit form of the Q -operator and fusion weights of the SOS model.

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1. Introduction

It is known that the (zero-field) eight-vertex model was explicitly solved by Baxter [3, 4] by the method of TQ -relation. Here the eight-vertex model is assumed with periodic boundary condition and *even* chain-size L (this restriction applies throughout this paper unless otherwise stated). In fact there are many Q -operators in this context [8], and the first one, discovered by Baxter in 1972 [4] (valid for both even and odd chain-site L), denoted by Q_{72} , was on the special 'root of unity' case

$$2N\eta_{72} = 2m_1K + im_2K', \quad i := \sqrt{-1}$$

where K, K' are the complete elliptic integrals, N, m_1, m_2 are integers, and η_{72} is the (crossing) parameter of the eight-vertex model. Using the Q_{72} -operator for $m_2 = 0$, Fabricius and McCoy computed the degeneracy of eight-vertex eigenvalues, a property previously found in [15, 16], then proposed the functional equations for the eight-vertex model at η_{72} in [23, 24] as an analogy with the set of functional equations known in the N -state chiral Potts model (CPM) [10]. The Fabricius–McCoy comparison between CPM and the eight-vertex model at η_{72}

was further analysed about their common mathematical structures in [29, 30], where the effort led to the discovery of Onsager-algebra symmetry of superintegrable CPM. However, by employing the Q_{72} -operator in the study of the special ‘root-of-unity’ eight-vertex model, the conjectural functional relations, strongly supported by computational evidences, are valid only in the cases for either even N or both N, m odd in the above η_{72} -expression ([23] (3.10), [24] (3.1), [26] (4.20)). Those cases do not include some ‘other’ important root-of-unity type of eight-vertex model that appeared in a sequence of 1973 papers [5–7] by Baxter in the study of the eight-vertex eigenvectors, where the parameter η satisfies the following *root of unity* condition ([5] (9), [6] (6.8), [7] (1.9) or [9] (113))¹:

$$\eta = \frac{2mK}{N}, \quad N \text{ and } m = \text{odd}, \quad \gcd(N, m) = 1. \quad (1.1)$$

In the present work, we study the eight-vertex models with the parameter η restricted only in the above case (1.1), which for convenience, will be loosely called the root-of-unity eight-vertex model throughout this paper. As noted in [23], the Q_{72} -operator and the Q -operator in [5–7] are different; in fact, it was shown in section II of [23] that Q_{72} -operator does not exist when η satisfies (1.1). Thus, the quest for a proper Q -operator in accordance with ‘symmetry’ of the eight-vertex model for the root-of-unity η in (1.1) appears to be a compelling problem for its solution.

The purpose of this paper is to construct a Q -operator of the eight-vertex model for the parameter η in (1.1), and to conduct the functional-relation study of the eight-vertex model as a parallel theory of the CPM. In the present paper, we provide a mathematical justification about the conjectural functional relations of the root-of-unity eight-vertex model by the explicit Q -operator constructed along the line, but not the same, as the Q_{72} -operator in [4]. Indeed, we produce the Q -operator by following the same mechanism in [32] of constructing the Q -operator of root-of-unity six-vertex at the N th root-of-unity anisotropic parameter q with odd N , which can be regarded as the vanishing elliptic nome limit of (1.1). Consequently, they share some remarkable qualitative and semi-quantitative resemblances in the functional-relation study of root-of-unity symmetry of the theory. Furthermore, our Q -operator (more precisely, the Q_R -operator) coincides with the eight-vertex Q -operator recently found by Fabricius in [28], but with the different specified values for the free parameter, subsequently a subtle difference occurs in the expression of Q -functional equation about the related involution (see formula (3) and section 3 in [28], and (2.19), (2.22) of this paper). One special character of the Q -operator in this work is that it possesses essential features appeared in Baxter’s 1973 papers [5–7], where he invented the original techniques to convert the eight-vertex model to an ice-type solid-on-solid (SOS) model, and derived the equation of eigenvectors by a generalized Bethe ansatz method. It is worth noting that the three original papers by Baxter in 1973 subsequently laid the foundation to many exceptional developments in the theory of quantum integrable systems, among which were the restricted SOS model [1], algebraic Bethe ansatz of the eight-vertex model [34], the theory of elliptic quantum group [18, 19], and the recently developed analytic theory of functional relations in the eight-vertex/SOS model in [11]. As the main observation of this work, we find a Q -operator of the root-of-unity eight-vertex model built upon the Baxter’s eigenvectors in [6] with parameters s, t taking certain special values so that results in [5–7] can be employed in calculations when verifying the relation between the N th-fusion operator and Q -operator, which serves as the Q -operator-constraint about the ‘root-of-unity’ symmetry of the theory. In this way, by assuming (as to be the case by numerical evidences for small L) the non-singular property of a certain $M(v_0)$ -matrix

¹ The letters N, L, m in this paper are the L, N, m_1 in [5–7]. Here for simplicity, we consider only the case $m_2 = 0$ in [5–7] by easier calculations of Jacobi theta functions (indeed no essential difficulties could arise for other cases by using the modified elliptic functions in [5] (10)).

(see (3.26) in the paper), the whole set of functional relations can be successfully justified, much as seen mathematically in [32] for a similar discussion of the root-of-unity six-vertex model. As a consequence, the conjectural Q -functional relation raised in [23, 24] is verified for the root-of-unity eight-vertex model with η in (1.1), but using a different involution appeared in the formula.

This paper is organized as follows. In section 2, we recall known results and conjectures in the root-of-unity eight-vertex model. We first briefly review some basic facts of the eight-vertex model in subsection 2.1. In subsection 2.2, we construct the fusion matrices from the fused L -operators, and derive the fusion relations of the eight-vertex model. In the root-of-unity cases, we discuss the relationship between TQ -, QQ - and Q -functional relations, in particular the equivalent relation between the QQ - and Q -functional relations. Section 3 contains the main results of this paper about the Q -operator and the functional relations in the root-of-unity eight-vertex model, as a parallel theory to the superintegrable CPM and root-of-unity six-vertex model established in [30, 32]. We first briefly summarize some main features in the eight-vertex SOS model and fusion weights in [5–7, 12, 13], needed for later discussions. In subsection 3.1, by imitating Baxter’s method of producing Q_{72} -operator in [4], we derive another Q -operator, different from Q_{72} , of the eight-vertex model with the parameter η only in (1.1). Using results known in the eight-vertex SOS model, we then in subsection 3.2 show the validity of functional relations using the constructed Q -operator. We close in section 4 with some concluding remarks.

2. Eight-vertex model and the fusion operators

We start with some basic facts about the eight-vertex model in subsection 2.1. Then in subsection 2.2, we construct the fusion matrix, and establish the fusion relations of the eight-vertex model; for the root-of-unity eight-vertex model, we discuss the relationship between TQ -, QQ - and Q -functional relations.

2.1. Formalism and quantum determinant of the eight-vertex model

First we review some basic notions in the eight-vertex model (for more details, see any standard reference listed in the biography, such as [8, 34] and references therein). This also serves to establish the notation.

The Boltzmann weights of the eight-vertex model are described by the homogeneous coordinates of an elliptic curve in the complex projective 3-space \mathbf{P}^3 :

$$E (= E_{\Delta, \gamma}): \quad a^2 + b^2 - c^2 - d^2 = 2\Delta(ab + cd), \quad cd = \gamma ab, \quad a : b : c : d \in \mathbf{P}^3 \quad (2.1)$$

where $\Delta, \gamma \in \mathbf{C}$. One can parameterize the above elliptic curve in terms of Jacobi theta functions of moduli $k, k' = (1 - k^2)^{1/2}$ with the complete elliptic integrals K, K' :

$$\begin{aligned} a &= \Theta(2\eta)\Theta(v - \eta)H(v + \eta) = \rho(v)\operatorname{sn}(v + \eta), \\ b &= \Theta(2\eta)H(v - \eta)\Theta(v + \eta) = \rho(v)\operatorname{sn}(v - \eta), \\ c &= H(2\eta)\Theta(v - \eta)\Theta(v + \eta) = \rho(v)\operatorname{sn}(2\eta), \\ d &= H(2\eta)H(v - \eta)H(v + \eta) = \rho(v)k \operatorname{sn}(2\eta)\operatorname{sn}(v - \eta)\operatorname{sn}(v + \eta), \end{aligned}$$

where $H(v) = \vartheta_1\left(\frac{v}{2K}, \tau\right)$, $\Theta(v) = \vartheta_4\left(\frac{v}{2K}, \tau\right)$, $\rho(v) = k^{\frac{1}{2}}\Theta(2\eta)\Theta(v - \eta)\Theta(v + \eta)$ with $\tau = \frac{iK'}{K}$, and the relations

$$H(v - 2K) = H(-v) = -H(v), \quad \Theta(v - 2K) = \Theta(-v) = \Theta(v). \quad (2.2)$$

Hence $H(v + 2N\eta) = H(v)$, $\Theta(v + 2N\eta) = \Theta(v)$ for η in (1.1). The parameters Δ, γ in (2.1) are given by $\Delta = \frac{\operatorname{cn}(2\eta)\operatorname{dn}(2\eta)}{1+k\operatorname{sn}^2(2\eta)}$, $\gamma = k\operatorname{sn}^2(2\eta)$, ((43), (44) and (54) in [5]). The uniformizing variable $v \in \mathbf{C}$ is called the spectral parameter. The elliptic curve E in (2.1) is biregular to the complex torus of \mathbf{C} quotiented by the lattice $4K\mathbf{Z} + 2K'\mathbf{Z}$: $E = \mathbf{C}/(4K\mathbf{Z} + 2K'\mathbf{Z})$. The quotient of E by the \mathbf{Z}_2 -automorphism, $v \mapsto v - 2K$, (corresponding to $a : b : c : d \mapsto -a : -b : c : d$), is the torus $\mathbf{C}/(2K\mathbf{Z} + 2K'\mathbf{Z})$. In this paper, we shall consider the order- N automorphism U of elliptic curve E with η in (1.1), and the elliptic function $h(v)$:

$$h(v) = \Theta(0)\Theta(v)H(v), \quad U : v \mapsto v - 2\eta. \tag{2.3}$$

Using the elliptic coordinates a, b, c, d in (2.1), one defines the L -operator of the eight-vertex model, which is the matrix of \mathbf{C}^2 -auxiliary and \mathbf{C}^2 -quantum space

$$L(v) = \begin{pmatrix} L_{0,0} & L_{0,1} \\ L_{1,0} & L_{1,1} \end{pmatrix} (v), \quad v \in \mathbf{C}, \tag{2.4}$$

where entries $L_{i,j}$ are the \mathbf{C}^2 -(quantum-space) operators

$$L_{0,0} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad L_{0,1} = \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix}, \quad L_{1,0} = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \quad L_{1,1} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}.$$

Equivalently to say, the eight-vertex weights are

$$\begin{aligned} R(+, +|+, +) &= R(-, -|- , -) = a, & R(+, +|- , -) &= R(-, -|+, +) = b, \\ R(+, -|- , +) &= R(-, +|+, -) = c, & R(+, -|+, -) &= R(-, +|- , +) = d. \end{aligned} \tag{2.5}$$

The L -operator (2.4) satisfies the YB relation,

$$R_{8v}(v' - v) \left(L(v') \underset{\text{aux}}{\otimes} 1 \right) \left(1 \underset{\text{aux}}{\otimes} L(v) \right) = \left(1 \underset{\text{aux}}{\otimes} L(v) \right) \left(L(v') \underset{\text{aux}}{\otimes} 1 \right) R_{8v}(v' - v), \tag{2.6}$$

with the R -matrix

$$R_{8v}(v) = \begin{pmatrix} a(v + \eta) & 0 & 0 & d(v + \eta) \\ 0 & b(v + \eta) & c(v + \eta) & 0 \\ 0 & c(v + \eta) & b(v + \eta) & 0 \\ d(v + \eta) & 0 & 0 & a(v + \eta) \end{pmatrix}$$

(see, e.g., [34]). Then the monodromy matrix of chain-size L ,

$$M(v) = L_1(v) \otimes \cdots \otimes L_L(v) = \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix}, \tag{2.7}$$

again satisfies the YB equation (2.6). The traces of monodromy matrices

$$T(v) := \operatorname{tr}_{\text{aux}} M(v) = A(v) + D(v), \quad v \in \mathbf{C}, \tag{2.8}$$

form a commuting family of $\overset{L}{\otimes} \mathbf{C}^2$ -operators, called the transfer matrix of the eight-vertex model, which commutes with the spin-reflection operator R and the S -operator:

$$[T(v), S] = [T(v), R] = 0, \quad \text{where } S = \prod_{\ell=1}^L \sigma_{\ell}^z, \quad R = \prod_{\ell=1}^L \sigma_{\ell}^x. \tag{2.9}$$

As the R -matrix R_{8v} at $v = -2\eta$ is of rank-one:

$$R_{8v}(-2\eta) = -\operatorname{sn}(2\eta) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the quantum determinant of (2.7) is defined by (2.6) with $v' = v - 2\eta$:

$$\begin{aligned} \det_q M(v) \cdot R_{8v}(-2\eta) &= R_{8v}(-2\eta) \left(M(v-2\eta) \otimes_{\text{aux}} 1 \right) \left(1 \otimes_{\text{aux}} M(v) \right) \\ &= \left(1 \otimes_{\text{aux}} M(v) \right) \left(M_L(v-2\eta) \otimes_{\text{aux}} 1 \right) R_{8v}(-2\eta). \end{aligned}$$

The above relation is equivalent to the following set of relations:

$$\begin{aligned} B(v)A(v-2\eta) &= A(v)B(v-2\eta), & D(v)C(v-2\eta) &= C(v)D(v-2\eta), \\ A(v-2\eta)C(v) &= C(v-2\eta)A(v), & B(v-2\eta)D(v) &= D(v-2\eta)B(v), \\ \det_q M(v) &= D(v)A(v-2\eta) - C(v)B(v-2\eta) = A(v)D(v-2\eta) - B(v)C(v-2\eta) \\ &= A(v-2\eta)D(v) - C(v-2\eta)B(v) = D(v-2\eta)A(v) - B(v-2\eta)C(v), \end{aligned} \tag{2.10}$$

with the explicit form of quantum determinant: $\det_q M(v) = h(v + \eta)^L h(v - 3\eta)^L$.

2.2. Fusion relation of the eight-vertex model

As in the six-vertex model [32], we now construct the eight-vertex fusion matrices $T^{(J)}(v)$ for a non-negative integer J from the J th fused L -operator $L^{(J)}(v)$, which is a matrix of \mathbf{C}^2 -quantum and \mathbf{C}^J -auxiliary space defined as follows. Denote the standard basis $|\pm 1\rangle$ of the \mathbf{C}^2 -auxiliary space of $L(v)$ in (2.4) by $\widehat{x} = |1\rangle, \widehat{y} = |-1\rangle$, and its dual basis by x, y . For non-negative integers m and n , $\widehat{x}^m \widehat{y}^n$ is the completely symmetric $(m+n)$ -tensor of \mathbf{C}^2 defined by

$$\binom{m+n}{n} \widehat{x}^m \widehat{y}^n = \underbrace{\widehat{x} \otimes \dots \otimes \widehat{x}}_m \otimes \underbrace{\widehat{y} \otimes \dots \otimes \widehat{y}}_n + \text{all other terms by permutations,}$$

similarly for $x^m y^n$. For $J \geq 1$, the \mathbf{C}^J -auxiliary space is the space of completely symmetric $(J-1)$ -tensors of \mathbf{C}^2 with the canonical basis $e_k^{(J)}$ and the dual basis $e_k^{(J)*}$:

$$e_k^{(J)} = \widehat{x}^{J-1-k} \widehat{y}^k, \quad e_k^{(J)*} = \binom{J-1-k}{k} x^{J-1-k} y^k, \quad k = 0, \dots, J-1. \tag{2.11}$$

By the first and third relations in (2.10) for $L = 1$, or equivalently,

$$\begin{aligned} \langle x^2 | L(v) \otimes_{\text{aux}} L(v-2\eta) | \widehat{x} \wedge \widehat{y} \rangle &= \langle y^2 | L(v) \otimes_{\text{aux}} L(v-2\eta) | \widehat{x} \wedge \widehat{y} \rangle = 0, \\ \langle x \otimes y | L(v) \otimes_{\text{aux}} L(v-2\eta) | \widehat{x} \wedge \widehat{y} \rangle &= \langle y \otimes x | L(v) \otimes_{\text{aux}} L(v-2\eta) | -\widehat{x} \wedge \widehat{y} \rangle = \frac{1}{2} \det_q L(v), \end{aligned}$$

where $\widehat{x} \wedge \widehat{y} = \frac{1}{2}(\widehat{x} \otimes \widehat{y} - \widehat{y} \otimes \widehat{x})$, the following relations hold:

$$\begin{aligned} \langle e_k^{(3)*} | L(v) \otimes_{\text{aux}} L(v-2\eta) | \widehat{x} \otimes \widehat{y} \rangle \\ = \langle e_k^{(3)*} | L(v) \otimes_{\text{aux}} L(v-2\eta) | \widehat{y} \otimes \widehat{x} \rangle \quad \text{for } k = 0, 1, 2. \end{aligned}$$

As a consequence for an integer $J \geq 2$, and $v_i = \widehat{x}$ or \widehat{y} for $1 \leq i \leq J-1$, we have

$$\begin{aligned} \langle e_k^{(J)*} | L(v) \otimes_{\text{aux}} \dots \otimes_{\text{aux}} L(v-2(J-3)\eta) \otimes_{\text{aux}} L(v-2(J-2)\eta) | v_1 \otimes \dots \otimes v_{J-1} \rangle \\ = \langle e_k^{(J)*} | L(v) \otimes_{\text{aux}} \dots \otimes_{\text{aux}} L(v-2(J-3)\eta) \\ \otimes_{\text{aux}} L(v-2(J-2)\eta) | v_{\sigma_1} \otimes \dots \otimes v_{\sigma_{J-1}} \rangle \end{aligned} \tag{2.12}$$

where $0 \leq k \leq J-1$, and σ is an arbitrary permutation of $\{1, \dots, J-1\}$. By using the basis (2.11) of the \mathbf{C}^J -auxiliary space, the fused $L^{(J)}$ -operator of eight-vertex model,

$L^{(J)}(v) = (L_{k,l}^{(J)}(v))_{0 \leq k,l \leq J-1}$, is defined by the following \mathbf{C}^2 -(quantum-space)-operators $L_{k,l}^{(J)}(v)$:

$$L_{k,l}^{(J)}(v) = \frac{\langle e_k^{(J)*} | L(v) \otimes_{\text{aux}} L(v-2\eta) \otimes_{\text{aux}} \cdots \otimes_{\text{aux}} L(v-2(J-2)\eta) | e_l^{(J)} \rangle}{\prod_{i=0}^{J-3} h(v-(2i+1)\eta)}. \quad (2.13)$$

For $J = 3$, by additive formulae of theta functions ([8] (15.4.25) (15.4.26)),

$$\begin{aligned} \Theta(u)\Theta(v)\Theta(a-u)\Theta(a-v) - H(u)H(v)H(a-u)H(a-v) \\ = \Theta(0)\Theta(a)\Theta(u-v)\Theta(a-u-v), \\ H(v)H(a-v)\Theta(u)\Theta(a-u) - \Theta(v)\Theta(a-v)H(u)H(a-u) \\ = \Theta(0)\Theta(a)H(v-u)H(a-u-v), \end{aligned}$$

$\langle e_k^{(3)*} | L(v) \otimes_{\text{aux}} L(v-2\eta) | e_l^{(3)} \rangle$ for $0 \leq k, l \leq 2$ are all divisible by $h(v-\eta)$. Indeed $L_{k,l}^{(3)} (= L_{k,l}^{(3)}(v))$ are expressed by

$$\begin{aligned} L_{0,0}^{(3)} &= \frac{\Theta(2\eta)^2}{\Theta(0)} \begin{pmatrix} H(v+\eta)\Theta(v-3\eta) & 0 \\ 0 & \Theta(v+\eta)H(v-3\eta) \end{pmatrix}, \\ &L_{0,0}^{(3)} \leftrightarrow L_{2,2}^{(3)}, \quad \Theta(v+j\eta) \leftrightarrow H(v+j\eta), \\ L_{1,0}^{(3)} &= H(4\eta) \begin{pmatrix} 0 & \Theta(v-\eta)^2 \\ H(v-\eta)^2 & 0 \end{pmatrix}, \\ &L_{1,0}^{(3)} \leftrightarrow L_{1,2}^{(3)}, \quad \Theta(v+j\eta) \leftrightarrow H(v+j\eta), \\ L_{2,0}^{(3)} &= \frac{H^2(2\eta)}{\Theta(0)} \begin{pmatrix} \Theta(v+\eta)H(v-3\eta) & 0 \\ 0 & H(v+\eta)\Theta(v-3\eta) \end{pmatrix}, \\ &L_{2,0}^{(3)} \leftrightarrow L_{0,2}^{(3)}, \quad \Theta(v+j\eta) \leftrightarrow H(v+j\eta), \\ L_{0,1}^{(3)} &= \frac{\Theta(2\eta)H(2\eta)}{\Theta(0)} \begin{pmatrix} 0 & H(v+\eta)H(v-3\eta) \\ \Theta(v+\eta)\Theta(v-3\eta) & 0 \end{pmatrix}, \\ &L_{0,1}^{(3)} \leftrightarrow L_{2,1}^{(3)}, \quad \Theta(v+j\eta) \leftrightarrow H(v+j\eta), \\ L_{1,1}^{(3)} &= \Theta(4\eta) \begin{pmatrix} H(v-\eta)\Theta(v-\eta) & 0 \\ 0 & \Theta(v-\eta)H(v-\eta) \end{pmatrix}. \end{aligned}$$

By this, all $L_{k,l}^{(J)}(v)$ in (2.12) are elliptic functions without poles ([13] lemma 2.3.1). Using $L^{(J)}(v)$ as the local operator, one defines the J th fusion matrix $T^{(J)}(v)$ as the trace of the monodromy matrix:

$$T^{(J)}(v) = \text{tr}_{\mathbf{C}^J} \left(\bigotimes_{\ell=1}^L L_{\ell}^{(J)}(v) \right), \quad L_{\ell}^{(J)}(v) = L^{(J)}(v) \text{ at site } \ell, \quad (2.14)$$

which form a family of commuting operators of the quantum space $\bigotimes^L \mathbf{C}^2$ with $T^{(2)}(v) = T(v)$. One can derive the recursive fusion relation among $T^{(J)}$'s as the case of six-vertex model (see, e.g., section 3 of [33]). Regard the auxiliary-space tensor $\mathbf{C}^2 \otimes \mathbf{C}^J$ as a subspace of $\bigotimes^{J+1} \mathbf{C}^2$, the auxiliary space \mathbf{C}^{J+1} as a subspace of $\mathbf{C}^2 \otimes \mathbf{C}^J$ by identifying the basis elements:

$$e_{k+1}^{(J+1)} = \frac{1}{\binom{J}{k+1}} \left(\binom{J-1}{k+1} \hat{x} \otimes e_{k+1}^{(J)} + \binom{J-1}{k} \hat{y} \otimes e_k^{(J)} \right), \quad k = -1, \dots, J-1.$$

Denote $f_k^{(J-1)} := \widehat{x} \otimes e_{k+1}^{(J)} - \widehat{y} \otimes e_k^{(J)}$ for $0 \leq k \leq J-2$. Then $e_l^{(J+1)}, f_k^{(J-1)}$ form a basis of $\mathbf{C}^2 \otimes \mathbf{C}^J$, with the dual basis $e_l^{(J+1)*}, f_k^{(J-1)*}$ expressed by

$$e_{k+1}^{(J+1)*} = x \otimes e_{k+1}^{(J)*} + y \otimes e_k^{(J)*},$$

$$f_k^{(J-1)*} = \frac{1}{\binom{J}{k+1}} \left(\binom{J-1}{k} x \otimes e_{k+1}^{(J)*} - \binom{J-1}{k+1} y \otimes e_k^{(J)*} \right).$$

One has

$$L_{k,l}^{(J+1)}(v) = \langle e_k^{(J+1)*} | L^{(J+1)}(v) | e_l^{(J+1)} \rangle$$

$$= \frac{1}{h(v - (2J - 3)\eta)} \langle e_k^{(J+1)*} | L^{(J)}(v) \otimes_{\text{aux}} L(v - 2(J - 1)\eta) | e_l^{(J+1)} \rangle,$$

$$\langle e_l^{(J+1)*} | L^{(J)}(v) \otimes_{\text{aux}} L(v - 2(J - 1)\eta) | f_k^{(J-1)} \rangle = 0,$$

$$\langle f_k^{(J-1)*} | L^{(J)}(v) \otimes_{\text{aux}} L(v - 2(J - 1)\eta) | f_l^{(J-1)} \rangle$$

$$= h(v - (2J - 1)\eta) \langle e_k^{(J-1)*} | L^{(J-1)}(v) | e_l^{(J-1)} \rangle.$$

Then follows the recursive fusion relation by setting $T^{(0)} = 0, T^{(1)} = h(v + \eta)^L$:

$$T^{(J)}(v)T^{(2)}(v - 2(J - 1)\eta) = h^L(v - (2J - 1)\eta)T^{(J-1)}(v)$$

$$+ h^L(v - (2J - 3)\eta)T^{(J+1)}(v), \quad J \geq 1. \tag{2.15}$$

Since the chain-size L is even, by (2.2) one finds the periodic property of $T^{(J)}$:

$$T(v - 2K) = T(v), \quad T^{(J)}(v - 2K) = T^{(J)}(v).$$

The eigenvalues of $T(v)$ are computed in [3, 4, 8] using an auxiliary Q -matrix, i.e., a commuting family, $Q(v)$ for $v \in \mathbf{C}$, with $[T(v), Q(v)] = 0$, and the Baxter's TQ -relation

$$T(v)Q(v) = h^L(v - \eta)Q(v + 2\eta) + h^L(v + \eta)Q(v - 2\eta),$$

equivalently,

$$T(v)Q(v) = \tilde{h}(v)^L Q(U^{-1}v) + \tilde{h}(U^{-1}v)^L Q(Uv), \quad \tilde{h}(v) := h(v - \eta), \tag{2.16}$$

where U is defined in (2.3). By (2.15) and (2.16) and using an induction argument, one finds the $T^{(J)}Q$ -relation for $J \geq 0$:

$$T^{(J)}(v) = Q(U^{-1}v)Q(U^{J-1}v) \sum_{k=0}^{J-1} (\tilde{h}(U^{k-1}v)^L Q(U^{k-1}v)^{-1} Q(U^k v)^{-1}). \tag{2.17}$$

In the root-of-unity case (1.1), the relation (2.17) in turn yields the boundary fusion relation:

$$T^{(N+1)}(v) = T^{(N-1)}(Uv) + 2\tilde{h}(U^{-1}v)^L. \tag{2.18}$$

Furthermore, it is expected that some Q -operator will encode essential features about the root-of-symmetry of the eight-vertex model, much as in the study of Onsager-algebra symmetry of superintegrable N -state CPM in [30]. Parallel to the functional equation of the chiral Potts transfer matrix ([10] (4.40)), the root-of-unity Q -operator of eight-vertex model conjecturally satisfies the Q -functional equation ([23] (3.10), [24](3.1)):

$$Q(Cv) = M_0 Q(v) \sum_{k=0}^{N-1} (\tilde{h}(U^k v)^L Q(U^k v)^{-1} Q(U^{k+1} v)^{-1}) \tag{2.19}$$

where C is an order-2 automorphism of elliptic curve E commuting with U , and M_0 is some normalized matrix independent of v . By (2.17), the Q -functional equation (2.19) is the same as the N th Q -relation ([24] (3.11), [30] (44), [32] theorem 3.1)

$$T^{(N)}(Uv) = M_0^{-1} Q(Cv)Q(v), \tag{2.20}$$

which is equivalent to either one of the following QQ -relations:

$$T^{(J)}(Uv) + T^{(N-J)}(U^{J+1}v) = M_0^{-1} Q(CU^J v) Q(v), \quad 0 \leq J \leq N. \quad (2.21)$$

In the next section, we are going to construct a Q -operator satisfying (2.20) with the elliptic automorphism C defined by²

$$C : v \mapsto v - 2K. \quad (2.22)$$

Remark. In the vanishing elliptic nome limit, $K \rightarrow \frac{\pi}{2}$, $K' \rightarrow \infty$, and the η in (1.1) tends to the N th root-of-unity q for odd N in the six-vertex model with the spectral parameter s as the limiting value of $-e^{\frac{\pi i v}{2K}}$. Then the automorphism (2.22) corresponds the s -involution, $C_0(s) = -s$, in the XXZ limit. The Q -operator of the six-vertex model at a such q , satisfying functional relations corresponding to (2.19)–(2.21) with the involution C_0 , was constructed in [32] theorem 4.2.

3. The Q -operator and verification of functional relations of the eight-vertex model at $\eta = \frac{2mK}{N}$

In this section, we discuss the functional relations of the eight-vertex model incorporated with the root-of-unity property. First we construct in subsection 3.1 the Q -operator of the eight-vertex model for the root-of-unity η (1.1) following the Baxter's method of producing Q_{72} in [4]. The Q -operator obtained here differs from Q_{72} , but will accord with the root-of-unity symmetry of the eight-vertex model in the sense that the set of functional relations is valid for this special Q -operator. Furthermore, the structure of the Q -operator can be fed into the scheme of Baxter's eight-vertex SOS model in the eight-vertex-eigenvector discussion in [5–7]. In subsection 3.2, we give a mathematical verification about the conjectured functional relations. The methods rely on results in [5–7] about the equivalence between the eight-vertex model and the SOS model, plus the study of fusion weights of the eight-vertex SOS model in [12, 13], which we now briefly summarize as follows.

For the convenience, we introduce the vectors and covectors as in [6] (6.4):

$$|v\rangle = \begin{pmatrix} H(v) \\ \Theta(v) \end{pmatrix}, \quad \langle v| = \begin{pmatrix} \Theta(v), & -H(v) \end{pmatrix}, \quad v \in \mathbf{C}. \quad (3.1)$$

For $s \in \mathbf{C}$ and integers $l \in \mathbf{Z}$, we define the local vector in [6] (C.13):

$$\Phi_{l,l+\mu} (= \Phi_{l,l+\mu}(v)) = |s + 2l\eta + \mu(\eta - v)\rangle, \quad \mu = \pm 1, \quad (3.2)$$

and the product-vector ([6] (3.2)) for a set of integers l_1, \dots, l_{L+1} with $l_{\ell+1} - l_\ell = \pm 1$ for $1 \leq \ell \leq L$:

$$\psi(l_1, \dots, l_{L+1})(v) = \Phi_{l_1, l_2} \otimes \Phi_{l_2, l_3} \otimes \cdots \otimes \Phi_{l_L, l_{L+1}}. \quad (3.3)$$

In the study of eigenvectors of the eight-vertex transfer matrices, Baxter converted it to a SOS model [5–7] so that the eight-vertex weights $R(\alpha, \beta | \lambda, \mu)$ (2.5) are changed to the SOS 'lattice-weights' $W(m, m' | l, l')$ by employing the local vectors $\Phi_{m,l}$ in (3.2) through the relation

$$\sum_{\beta, \mu} R(\alpha, \beta | \lambda, \mu)(v) \Phi_{l, l'}(v')_{\beta} z_{m', l'}(v, v')_{\mu} = \sum_m W(m, m' | l, l')(v) \Phi_{m, m'}(v')_{\alpha} z_{m, l}(v, v')_{\lambda}, \quad (3.4)$$

² Note that the automorphism C here differs from the $C_{72} : v \mapsto v - iK'$ for the Q_{72} -operator in [24]. Hence the Q -operator in this paper carries a different nature from the Baxter's Q_{72} in [4].

where $\alpha, \beta, \lambda, \mu = \pm 1, l, l', m, m' \in \mathbf{Z}$ with $|l - l'| = |m - m'| = |m - l| = |m' - l'| = 1$, and $z_{m,l}$ are the vectors³

$$z_{l+1,l}(v, v') = |s + v - v' + 2l\eta\rangle, \quad z_{l-1,l}(v, v') = |s - v + v' + 2l\eta\rangle.$$

Indeed, the Boltzmann weights $W(m, m'|l, l')$ for integers m, m', l, l' are zeros except $\lambda = m - l, \mu = m' - l', \alpha = m' - m, \beta = l' - l$ equal to ± 1 , and the nonzero weights are

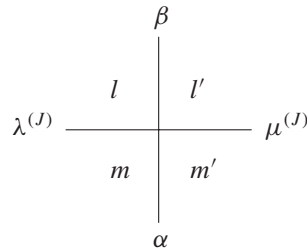
$$\begin{aligned} W(l \pm 1, l \pm 2|l, l \pm 1)(v) &= \frac{h(v + \eta)}{h(2\eta)}, & (\lambda = \mu = \alpha = \beta = \pm 1), \\ W(l \pm 1, l|l, l \mp 1)(v) &= \frac{h(s - K + 2(l \mp 1)\eta)h(v - \eta)}{h(2\eta)h(s - K + 2l\eta)}, & (\lambda = \mu = -\alpha = -\beta = \pm 1), \\ W(l \pm 1, l|l, l \pm 1)(v) &= \frac{h(s - K + 2l\eta \mp (v - \eta))}{h(s - K + 2l\eta)}, & (\lambda = -\mu = -\alpha = \beta = \pm 1), \end{aligned} \tag{3.5}$$

((C. 14) and (C30)–(C.33) of [6], or (2.1.4a)–(2.1.4c) of [13]⁴). In the case (1.1), by using (3.4), the eight-vertex transfer matrix (2.8) transfers the vector $\psi(l_1, \dots, l_{L+1})$ in (3.3) to a linear combination of product-vectors:

$$T(v)\psi(l_1, \dots, l_{L+1})(v') = \sum_{m_\ell} \left\{ \prod_{\ell=1}^L W(m_\ell, m_{\ell+1}|l_\ell, l_{\ell+1})(v) \right\} \psi(m_1, \dots, m_{L+1})(v'), \tag{3.6}$$

where the summation runs over integers m_ℓ 's with $m_{\ell+1} = m_\ell \pm 1, m_\ell = l_\ell \pm 1$ for all ℓ , and $l_{L+1} - l_1 = m_{L+1} - m_1 \equiv 0 \pmod{N}$ ((1.5), (1.9) and (1.11) of [7]).

For a positive integer J , one can derive the J th-fusion weights, $W^{(J)}(m, m'|l, l')(v)$, of the SOS model such that $W^{(2)}(m, m'|l, l')(v) = W(m, m'|l, l')(v)$ in (3.5). Indeed, $W^{(J)}(m, m'|l, l')(v)$ are zeros except $\lambda^{(J)}, \mu^{(J)} = J - 1 - 2k$ for $0 \leq k \leq J - 1$, and $|\alpha| = |\beta| = 1$, where $\lambda^{(J)} = m - l, \mu^{(J)} = m' - l', \lambda = m' - m, \mu = l' - l$:



By formulae (2.1.16), (2.1.20) in [13]⁵, the nonzero J th-fusion weights are given by

$$\begin{aligned} W^{(J)}(l - 1, l|l' - 1, l')(v) &= \frac{h(s - K + (l + l' + J - 1)\eta)h(v + (l - l' - J + 2)\eta)}{h(2\eta)h(s - K + 2l\eta)} \\ &= \frac{h(s - K + (2l - \mu^{(J)} + J - 1)\eta)h(v + (\mu^{(J)} - J + 2)\eta)}{h(2\eta)h(s - K + 2l\eta)}, \\ &(\lambda^{(J)} = \mu^{(J)}, \alpha = \beta = 1), \\ W^{(J)}(l + 1, l|l' + 1, l')(v) &= \frac{h(s - K + (l + l' - J + 1)\eta)h(v + (l' - l - J + 2)\eta)}{h(2\eta)h(s - K + 2l\eta)} \end{aligned}$$

³ The $z_{l-1,l}$ here is in [6] (B.26) (C.13), which differs from [6] (6.5) by a factor.
⁴ The $W_{11}(a, b, c, d), u, \lambda, \xi$ in [13] are equal to $W(a, b|d, c), \frac{v-\eta}{2\eta}, 2\eta, \frac{s-K}{2\eta}$ here, respectively.
⁵ The $W^{(J)}(m, m'|l, l')(v)$ here is equal to $W_{1,J-1}(m, m', l', l|u)$ in [13].

$$\begin{aligned}
&= \frac{h(s - K + (2l - \mu^{(J)} - J + 1)\eta)h(v - (\mu^{(J)} + J - 2)\eta)}{h(2\eta)h(s - K + 2l\eta)}, \\
&\quad (\lambda^{(J)} = \mu^{(J)}, \alpha = \beta = -1), \\
W^{(J)}(l + 1, l|l' - 1, l')(v) &= \frac{h((l' - l + J - 1)\eta)h(s - K + (l + l' + J - 2)\eta - v)}{h(2\eta)h(s - K + 2l\eta)} \\
&= \frac{h((-\mu^{(J)} + J - 1)\eta)h(s - K + (2l - \mu^{(J)} + J - 2)\eta - v)}{h(2\eta)h(s - K + 2l\eta)}, \\
&\quad (\lambda^{(J)} = \mu^{(J)} + 2, \alpha = -\beta = -1), \\
W^{(J)}(l - 1, l|l' + 1, l')(v) &= \frac{h((l - l' + J - 1)\eta)h(s - K + (l + l' - J + 2)\eta + v)}{h(2\eta)h(s - K + 2l\eta)} \\
&= \frac{h((\mu^{(J)} + J - 1)\eta)h(s - K + (2l - \mu^{(J)} - J + 2)\eta + v)}{h(2\eta)h(s - K + 2l\eta)}, \\
&\quad (\lambda^{(J)} = \mu^{(J)} - 2, \alpha = -\beta = 1). \tag{3.7}
\end{aligned}$$

For the N th ‘root-of-unity’ η in (1.1), as in (3.6), the J th fusion matrix $T^{(J)}$ in (2.14) is related to weights $W^{(J)}(m, m'|l, l')$ through product-vectors in (3.3) by

$$T^{(J)}(v)\psi(l_1, \dots, l_{L+1})(v') = \sum_{m_\ell} \left\{ \prod_{\ell=1}^L W^{(J)}(m_\ell, m_{\ell+1}|l_\ell, l_{\ell+1})(v) \right\} \psi(m_1, \dots, m_{L+1})(v'), \tag{3.8}$$

the summation being integers m_ℓ 's with $m_{\ell+1} = m_\ell \pm 1$, $m_\ell - l_\ell = J - 1 - 2k_\ell$, $0 \leq k_\ell \leq J - 1$, for all ℓ , and $l_{L+1} - l_1 = m_{L+1} - m_1 \equiv 0 \pmod{N}$ ([12, 13] theorem 2.3.3). Later in this paper, we shall work on $T^{(N)}$ with the variable evaluating at $v - \eta$, then the corresponding SOS weights in (3.7) become

$$\begin{aligned}
W^{(N)}(l - 1, l|l' - 1, l')(v - 2\eta) &= \frac{h(-K + (2l - \mu^{(N)} - 1)\eta)h(v + \mu^{(N)}\eta)}{h(2\eta)h(-K + 2l\eta)}, \\
&\quad (\lambda^{(N)} = \mu^{(N)}, \alpha = \beta = 1), \\
W^{(N)}(l + 1, l|l' + 1, l')(v - 2\eta) &= \frac{h(-K + (2l - \mu^{(N)} + 1)\eta)h(v - \mu^{(N)}\eta)}{h(2\eta)h(-K + 2l\eta)}, \\
&\quad (\lambda^{(N)} = \mu^{(N)}, \alpha = \beta = -1), \\
W^{(N)}(l + 1, l|l' - 1, l')(v - 2\eta) &= \frac{h((-\mu^{(N)} - 1)\eta)h(-K + (2l - \mu^{(N)})\eta - v)}{h(2\eta)h(-K + 2l\eta)}, \\
&\quad (\lambda^{(N)} = \mu^{(N)} + 2, \alpha = -\beta = -1), \\
W^{(N)}(l - 1, l|l' + 1, l')(v - 2\eta) &= \frac{h((\mu^{(N)} - 1)\eta)h(-K + (2l - \mu^{(N)})\eta + v)}{h(2\eta)h(-K + 2l\eta)}, \\
&\quad (\lambda^{(N)} = \mu^{(N)} - 2, \alpha = -\beta = 1). \tag{3.9}
\end{aligned}$$

Here we use the relation $h(v + N\eta) = -h(v)$ by the condition (1.1) on η .

3.1. The Q -operator of the eight-vertex model for $\eta = \frac{2mK}{N}$

As the construction of Q_{72} in [4], we start with the \mathbb{S} -, $\widehat{\mathbb{S}}$ -operator of \mathbb{C}^N -auxiliary and \mathbb{C}^2 -quantum space, $\mathbb{S} = (\mathbb{S}_{i,j})_{i,j \in \mathbb{Z}_N}$, $\widehat{\mathbb{S}} = (\widehat{\mathbb{S}}_{i,j})_{i,j \in \mathbb{Z}_N}$, where $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$, and $\mathbb{S}_{i,j}$, $\widehat{\mathbb{S}}_{i,j}$ are

\mathbf{C}^2 -operators. Here the \mathbf{C}^N -basis of the auxiliary space is indexed by \mathbf{Z}_N . The general forms of Q_R, Q_L -matrices are

$$Q_R = \text{tr}_{\mathbf{C}^N} \left(\bigotimes_{\ell=1}^L S_\ell \right), \quad Q_L = \text{tr}_{\mathbf{C}^N} \left(\bigotimes_{\ell=1}^L \widehat{S}_\ell \right), \quad \text{where } S_\ell, \widehat{S}_\ell = S, \widehat{S} \text{ at site } \ell, \quad (3.10)$$

with $TQ_R = \text{tr}_{\mathbf{C}^2 \otimes \mathbf{C}^N} (\bigotimes_{\ell=1}^L U_\ell)$, $Q_L T = \text{tr}_{\mathbf{C}^2 \otimes \mathbf{C}^N} (\bigotimes_{\ell=1}^L \widehat{U}_\ell)$, where $U_\ell, \widehat{U}_\ell = U, \widehat{U}$ at site ℓ , and U, \widehat{U} are the matrices of $\mathbf{C}^2 \otimes \mathbf{C}^N$ -auxiliary and \mathbf{C}^2 -quantum space

$$U = \begin{pmatrix} L_{0,0}S & L_{0,1}S \\ L_{1,0}S & L_{1,1}S \end{pmatrix}, \quad \widehat{U} = \begin{pmatrix} \widehat{S}L_{0,0} & \widehat{S}L_{0,1} \\ \widehat{S}L_{1,0} & \widehat{S}L_{1,1} \end{pmatrix}.$$

The operator $TQ_R, Q_L T$ will decompose into the sum of two matrices if we can find a $2N$ by $2N$ scalar matrix M (independent of v) such that

$$M^{-1}UM = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad M^{-1}\widehat{U}M = \begin{pmatrix} \widehat{A} & 0 \\ \widehat{C} & \widehat{D} \end{pmatrix}. \quad (3.11)$$

The above-required form is unaffected by postmultiplying M by an upper blocktriangular matrix. Together with a similar transformation of S , we can in general choose

$$M = \begin{pmatrix} I_N & \delta \\ 0 & I_N \end{pmatrix}, \quad \delta = \text{dia}[\delta_0, \dots, \delta_{N-1}]. \quad (3.12)$$

Hence

$$M^{-1}UM = \begin{pmatrix} L_{0,0}S - \delta L_{1,0}S, & L_{0,0}S\delta - \delta L_{1,0}S\delta + L_{0,1}S - \delta L_{1,1}S \\ L_{1,0}S, & L_{1,0}S\delta + L_{1,1}S \end{pmatrix}.$$

The condition for nonzero $S_{i,j}$'s in above with vanishing upper blocktriangular matrix is ([4] (C.10))

$$\begin{pmatrix} a\delta_j - \delta_i b, & d - c\delta_i\delta_j \\ c - d\delta_i\delta_j, & b\delta_j - \delta_i a \end{pmatrix} S_{i,j} = 0, \quad i, j \in \mathbf{Z}_N,$$

which in turn yields

$$(a^2 + b^2 - c^2 - d^2)\delta_i\delta_j = ab(\delta_i^2 + \delta_j^2) - cd(1 + \delta_i^2\delta_j^2). \quad (3.13)$$

If we set $\delta_i = k^{\frac{1}{2}} \text{sn}(u)$, by (2.1), then follows: $\delta_j = k^{\frac{1}{2}} \text{sn}(u \pm 2\eta)$. For $\delta_i = k^{\frac{1}{2}} \text{sn}(u)$, $\delta_j = k^{\frac{1}{2}} \text{sn}(u \pm 2\eta)$, using the general formulae

$$\begin{aligned} \text{sn } A \text{ sn } B - \text{sn } C \text{ sn } D &= \frac{\Theta(0)\Theta(A+B)H(A-D)H(B-D)}{k\Theta(A)\Theta(B)\Theta(C)\Theta(D)}, \\ 1 - k^2 \text{sn } A \text{ sn } B \text{ sn } C \text{ sn } D &= \frac{\Theta(0)\Theta(A+B)\Theta(A-D)\Theta(B-D)}{\Theta(A)\Theta(B)\Theta(C)\Theta(D)}, \end{aligned}$$

when $A + B = C + D$, one can derive

$$S_{i,j}(v) = S_{i,j}(u) = |u \pm (\eta - v)\tau_{i,j}, \quad \tau_{i,j} = (\tau_{i,j}^1, \tau_{i,j}^2),$$

and the relations

$$\begin{aligned} (L_{0,0}S_{i,j})(v) - \delta_i(L_{1,0}S_{i,j})(v) &= h(v - \eta) \frac{\Theta(u \pm 2\eta)}{\Theta(u)} S_{i,j}(v + 2\eta), \\ \delta_j(L_{1,0}S_{i,j})(v) + (L_{1,1}S_{i,j})(v) &= h(v + \eta) \frac{\Theta(u)}{\Theta(u \pm 2\eta)} S_{i,j}(v - 2\eta), \end{aligned}$$

where $h(v)$ is in (2.3). We choose s so that $\delta_i = k^{\frac{1}{2}} \operatorname{sn}(s + 2i\eta)$, $i \in \mathbf{Z}_N$, are N -distinct diagonal entries of δ in (3.12). Then⁶

$$S_{i,j}(v) = \begin{cases} |s + 2i\eta + (j - i)(\eta - v)|\tau_{i,j} & \text{if } j - i = \pm 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.14)$$

with the arbitrary parameters $\tau_{i,j}$. The A, D in (3.11) are related to S by

$$A(v) = h(v - 2\eta)d^{-1}S(v + 2\eta)d, \quad D(v) = h(v + 2\eta)dS(v - 2\eta)d^{-1},$$

where d is the diagonal matrix $\operatorname{dia}[\Theta(s), \Theta(s + 2\eta), \dots, \Theta(s + 2(N - 1)\eta)]$. This implies

$$T(v)Q_R(v) = h(v - 2\eta)^L Q_R(v + 2\eta) + h(v + 2\eta)^L Q_R(v - 2\eta).$$

With the similar discussion for $M^{-1}\widehat{U}M$, the nonzero $\widehat{S}_{i,j}$ again yields the relation (3.13). For $\delta_i = k^{\frac{1}{2}} \operatorname{sn}(u)$, $\delta_j = k^{\frac{1}{2}} \operatorname{sn}(u \pm 2\eta)$, one arrives the expression

$$\widehat{S}_{i,j}(= \widehat{S}_{i,j}(v)) = \widehat{\tau}_{i,j}|u \pm (\eta + v)|, \quad \widehat{\tau}_{i,j} = (\widehat{\tau}_{i,j;1}, \widehat{\tau}_{i,j;2})^t,$$

and the relations

$$\begin{aligned} (\widehat{S}_{i,j}L_{0,0})(v) - \delta_i(\widehat{S}_{i,j}L_{1,0})(v) &= h(v + \eta) \frac{\Theta(u \pm 2\eta)}{\Theta(u)} \widehat{S}_{i,j}(v - 2\eta), \\ \delta_j(\widehat{S}_{i,j}L_{1,0})(v) + (\widehat{S}_{i,j}L_{1,1})(v) &= h(v - \eta) \frac{\Theta(u)}{\Theta(u \pm 2\eta)} \widehat{S}_{i,j}(v + 2\eta). \end{aligned}$$

We define the δ in (3.12) by N -distinct numbers $\delta_i = k^{\frac{1}{2}} \operatorname{sn}(\widehat{s} + 2i\eta)$, $i \in \mathbf{Z}_N$, for some \widehat{s} . Then $\widehat{S}_{i,j}$ are given by

$$\widehat{S}_{i,j}(v) = \begin{cases} \widehat{\tau}_{i,j}|\widehat{s} + 2i\eta + (j - i)(\eta + v)| & \text{if } j - i = \pm 1, \\ 0 & \text{otherwise,} \end{cases} \quad (3.15)$$

and related to \widehat{A}, \widehat{D} in (3.11) by

$$\widehat{A}(v) = h(v + 2\eta)\widehat{d}^{-1}\widehat{S}(v - 2\eta)\widehat{d}, \quad \widehat{D}(v) = h(v - 2\eta)\widehat{d}\widehat{S}(v + 2\eta)\widehat{d}^{-1},$$

where $\widehat{d} = \operatorname{dia}[\Theta(\widehat{s}), \Theta(\widehat{s} + 2\eta), \dots, \Theta(\widehat{s} + 2(N - 1)\eta)]$. Hence

$$Q_L(v)T(v) = h(v - 2\eta)^L Q_L(v + 2\eta) + h(v + 2\eta)^L Q_L(v - 2\eta).$$

To construct a $Q(v)$ matrix from the Q_R - and Q_L -operator, as in [4] (C28) it suffices to find \widehat{s}, s in (3.15), (3.14) so that

$$Q_L(u)Q_R(v) = Q_L(v)Q_R(u), \quad u, v \in \mathbf{C}, \quad (3.16)$$

then define

$$Q(v) = Q_R(v)Q_R(v_0)^{-1} = Q_L(v_0)^{-1}Q_L(v), \quad (3.17)$$

where v_0 is a fixed value of v so that $Q_R(v_0)$ and $Q_L(v_0)$ are non-singular. Then $Q(v)$'s form a commuting family and satisfy the TQ -relation (2.16) ([4] (C28) (C37) (C38)). Note that by (3.17), the operator $Q(v)$, defined up to a normalized factor determined by v_0 , is independent of the choice of parameters $\tau_{i,j}, \widehat{\tau}_{i,j}$, regardless of the dependence of Q_R, Q_L on $\tau_{i,j}, \widehat{\tau}_{i,j}$.

Lemma 3.1. *The relation (3.16) is valid for $(\widehat{s}, s) = (2K, 0), (0, 2K)$ in (3.15), (3.14).*

⁶ The $S_{i,j}$ of (3.14) here is equal to the $(\widehat{S}_R)_{k,l}$ of (22) in [28], where k, l, L and the parameter t correspond to $i + 1, j + 1, N$ and $s - \eta$ in this paper. However in the discussion of Q_L -operator, there is a slight difference about the construction. Indeed, the $\widehat{S}_{i,j}$ of (3.15) in this paper differs from the $(\widehat{S}_L)_{k,l}$ in [28] (28) by the multiplication of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\widehat{S}_{i,j} = (\widehat{S}_L)_{k,l} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, with the identification of parameters: $\widehat{s} = t + \eta$.

Proof. Define the functions

$$g(v) := H(v)\Theta(v) \left(= \frac{1}{\Theta(0)}h(v) \right), \quad f(v) := \frac{2h(v-K)}{h(K)}.$$

Using the general formula ([6] (C.27)),

$$\Theta(2A)H(2B) - H(2A)\Theta(2B) = f(A+B)g(A-B), \tag{3.18}$$

one finds the following identity for $\widehat{s}, s \in \mathbf{C}$ and $\lambda, \mu = \pm 1$:

$$\begin{aligned} (\widehat{s} + 2i\eta + \lambda(\eta + u)|s + 2i'\eta + \mu(\eta - v)) &= f\left(\frac{\widehat{s} + s}{2} + \left(i + i' + \frac{\lambda + \mu}{2}\right)\eta + \frac{\lambda u - \mu v}{2}\right) \\ &\times g\left(\frac{\widehat{s} - s}{2} + \left(i - i' + \frac{\lambda - \mu}{2}\right)\eta + \frac{\lambda u + \mu v}{2}\right). \end{aligned} \tag{3.19}$$

By using (3.19), the product of $\widehat{S}_{i,j}(u), S_{i',j'}(v)$ in (3.15), (3.14) is expressed by

$$\widehat{S}_{i,j}(u)S_{i',j'}(v) = \widehat{\tau}_{i,j}\tau_{i',j'}f(u, v|i, j; i', j')g(u, v|i, j; i', j'),$$

where

$$\begin{aligned} f(u, v|i, j; i', j') &:= f\left(\frac{\widehat{s} + s}{2} + \left(i + i' + \frac{j - i + j' - i'}{2}\right)\eta + \frac{(j - i)u - (j' - i')v}{2}\right), \\ g(u, v|i, j; i', j') &:= g\left(\frac{\widehat{s} - s}{2} + (i - i')\eta + \frac{(j - i - j' + i')\eta}{2} + \frac{(j - i)u + (j' - i')v}{2}\right). \end{aligned}$$

The relation (3.16) will hold if there exist auxiliary functions $P(v, u|n), p(v, u|n)$ for $n \in \mathbf{Z}_N$ such that

$$\widehat{S}_{i,j}(u)S_{i',j'}(v) = p(v, u|i' + i)P(v, u|i' - i)\widehat{S}_{i,j}(v)S_{i',j'}(u)P(v, u|j' - j)^{-1}p(v, u|j' + j)^{-1}, \tag{3.20}$$

since the product $\widehat{S}_{i,j}(u)S_{i',j'}(v)$ differs only by a diagonal gauge transformation when interchanging v and u . The condition (3.20) is equivalent to the following relations:

$$\begin{aligned} g(u, v|i, j; i', j') &= P(v, u|i' - i)g(v, u|i, j; i', j')P(v, u|j' - j)^{-1}, \\ j - i &= -(j' - i') = \pm 1, \\ f(u, v|i, j; i', j') &= p(v, u|i' + i)f(u, v|i, j; i', j')p(v, u|j' + j)^{-1}, \\ j - i &= j' - i' = \pm 1. \end{aligned}$$

The conditions on $g(u, v|i, j; i', j')$ yield just one condition for P :

$$\frac{P(v, u|n+2)}{P(v, u|n)} = \frac{g\left(\frac{\widehat{s}-s}{2} - (n+1)\eta + \frac{u-v}{2}\right)}{g\left(\frac{\widehat{s}-s}{2} - (n+1)\eta + \frac{v-u}{2}\right)} \quad \text{for } n \in \mathbf{Z}_N.$$

Since N is odd, the constraint $P(v, u|n+2N) = P(v, u|n)$ in turn yields $\widehat{s} - s = \pm 2K$. Similarly, the conditions on $f(u, v|i, j; i', j')$ yields the condition on p :

$$\frac{p(v, u|n)}{p(v, u|n+2)} = \frac{f\left(\frac{\widehat{s}+s}{2} + (n+1)\eta + \frac{u-v}{2}\right)}{f\left(\frac{\widehat{s}+s}{2} + (n+1)\eta + \frac{v-u}{2}\right)} \quad \text{for } n \in \mathbf{Z}_N,$$

with $\widehat{s} + s = \pm 2K$. Then follows the values of \widehat{s}, s : $(\widehat{s}, s) = (\pm 2K, 0), (0, \pm 2K)$. Since $\widehat{S}_{i,j}, S_{i,j}$ are the same when adding $\pm 4K$ on \widehat{s}, s -values, only $(2K, 0), (0, 2K)$ remains as solutions of (\widehat{s}, s) in our discussion. \square

Remark. In the above proof, we indeed show the values of \widehat{s}, s in lemma 3.1 are the only solutions such that the relation (3.20) holds.

By lemma 3.1, there are only two sets of s, \widehat{s} -values in the discussion of Q -operator (3.17) of the root-of-unity eight-vertex model. Moreover these two Q -operators can be converted from one to another by the substitution of variables: $v \mapsto v - 2K$. So we need only to consider the Q -operator (3.17) with Q_R, Q_L using the parameters

$$s = 0, \widehat{s} = 2K.$$

Hence the $S_{i',j'}, \widehat{S}_{i,j}$ in (3.14) and (3.15) are zeros except $j - i = \pm 1$, in which cases⁷

$$S_{i,j}(v) = |2i\eta + (j - i)(\eta - v)\rangle\tau_{i,j}, \quad \widehat{S}_{i,j}(v) = \widehat{\tau}_{i,j}\langle 2K + 2i\eta + (j - i)(\eta + v)|. \quad (3.21)$$

The S -operator of the \mathbf{C}^N -auxiliary and \mathbf{C}^2 -quantum space is now with the form

$$S = (S_{i,j})_{i,j \in \mathbf{Z}_N} = \begin{pmatrix} 0 & S_{0,1} & 0 & \cdots & 0 & S_{0,N-1} \\ S_{1,0} & 0 & S_{1,2} & \ddots & & \vdots \\ 0 & & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & & & \ddots & & S_{N-2,N-1} \\ S_{N-1,0} & 0 & \cdots & 0 & S_{N-1,N-2} & 0 \end{pmatrix},$$

similarly for \widehat{S} . By (2.2), the Q -operator satisfies

$$Q(v - 2K) = SQ(v) = Q(v)S, \quad (3.22)$$

where S is the operator in (2.9).

3.2. Mathematical verification of functional relations in the eight-vertex model for $\eta = \frac{2mK}{N}$

Hereafter we set the parameter $(s, \widehat{s}) = (0, 2K)$ in Q_R, Q_L and local vectors in (3.2); so $S_{i',j'}, \widehat{S}_{i,j}$ are given by (3.21). In this subsection, we show the following theorem about functional relations of the root-of-unity eight-vertex model with η in (1.1) by using results in the eight-vertex SOS model [5–7, 12, 13].

Theorem 3.1. *The Q -operator (3.17) defined by (3.21) satisfies the Q -functional equation (2.19), equivalent to the QQ -relations, (2.20) or (2.21), with C in (2.22).*

Associated with the local vector $\Phi_{l,l+\mu}$ in (3.2) with $s = 0$, we introduce the local covector $\widehat{\Phi}_{l,l+\mu}$ for $l \in \mathbf{Z}, \mu = \pm 1$:

$$\begin{aligned} \Phi_{l,l+\mu} (= \Phi_{l,l+\mu}(v)) &= |2l\eta + \mu(\eta - v)\rangle, \\ \widehat{\Phi}_{l,l+\mu} (= \widehat{\Phi}_{l,l+\mu}(v)) &= r_{l,l+\mu}\langle 2K + 2l\eta + \mu(\eta + v)|, \quad r_{l,l+\mu} = \frac{\mu\Theta(0)h(K)}{2h(2\eta)h(2(l + \mu)\eta - K)}. \end{aligned} \quad (3.23)$$

By (3.19), one finds

$$\begin{pmatrix} \widehat{\Phi}_{l-1,l} \\ \widehat{\Phi}_{l+1,l} \end{pmatrix} (2K + v) \cdot (\Phi_{l,l+1}, \Phi_{l,l-1})(v) = \frac{h(v - \eta)}{h(2\eta)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

⁷ Note that the free parameter of the Q_R -operator in this paper specifies at a value different from that in [28] section 3. In this work, the values of s, \widehat{s} for the Q_R, Q_L -operators are distinct, unlike that in [28] (31) where the free parameter t for Q_R, Q_L takes the same value as in [8] (9.8.38)–(9.8.42) (also see the argument in [28] section 4). Indeed, the equality (31) in [28] (with the same t -value) provides only a sufficient condition for the commutativity of the constructed Q -operator, which is still valid when the relation (3.16) in this paper holds for two (not necessary equal) s, \widehat{s} -values.

For a set of integers l_1, \dots, l_{L+1} with $l_{\ell+1} - l_\ell = \pm 1$ for $1 \leq \ell \leq L$, the product-vector in (3.3) (for $s = 0$), and the product-covector are expressed by

$$\begin{aligned} \psi(l_1, \dots, l_{L+1})(v) &= \Phi_{l_1, l_2}(v) \otimes \Phi_{l_2, l_3}(v) \otimes \dots \otimes \Phi_{l_L, l_{L+1}}(v), \\ \widehat{\psi}(l_1, \dots, l_{L+1})(v) &= \widehat{\Phi}_{l_1, l_2}(v) \otimes \widehat{\Phi}_{l_2, l_3}(v) \otimes \dots \otimes \widehat{\Phi}_{l_L, l_{L+1}}(v). \end{aligned} \tag{3.24}$$

The condition (1.1) guarantees the local vector and covector in (3.23) unchanged when replacing l by $l \pm N$. Hence for $i, j \in \mathbf{Z}_N$ and $j - i = \pm 1$, we shall also write

$$\Phi_{i,j} := \Phi_{l, l+(j-i)} (= \Phi_{l'+i-j, l'}), \quad \widehat{\Phi}_{i,j} := \widehat{\Phi}_{l, l+(j-i)} (= \widehat{\Phi}_{l'+i-j, l'})$$

where $l \equiv i$ (or $l' \equiv j$) (mod N) if no confusion will arise. Similarly, one can write $\psi(i_1, \dots, i_{L+1})(v) = \psi(l_1, \dots, l_{L+1})(v)$, $\widehat{\psi}(i_1, \dots, i_{L+1})(v) = \widehat{\psi}(l_1, \dots, l_{L+1})(v)$ for $i_\ell \in \mathbf{Z}_N$ and $i_{\ell+1} - i_\ell = \pm 1$ ($1 \leq \ell \leq L$) using an integer-representative l_1 (or l_{L+1}) of i_1 (i_{L+1} resp.) in \mathbf{Z}_N . It is expected that product-vectors (covectors) in (3.24) for all l_ℓ 's and v span the 2^L -dimensional vector space $\otimes^L \mathbf{C}^2$ (the dual space $\otimes^L \mathbf{C}^{*2}$ resp.). Equivalently, product-vectors $\psi(l_1, \dots, l_{L+1})(v_0)$ for $l_1 \in \mathbf{Z}_N$ and $l_{\ell+1} - l_\ell = \pm 1$, span the vector space $\otimes^L \mathbf{C}^2$ for a generic v_0 ; the same for $\otimes^L \mathbf{C}^{*2}$ spanned by $\widehat{\psi}(l_1, \dots, l_{L+1})(v_0)$'s. We shall use v_0 to denote a general value of v . For simple notations, we shall also write these vectors at v_0 in $\otimes^L \mathbf{C}^2$ and $\otimes^L \mathbf{C}^{*2}$ by

$$\begin{aligned} v(l_1, \dots, l_{L+1}) &:= \psi(l_1, \dots, l_{L+1})(v_0), \\ h(l_1, \dots, l_{L+1}) &:= \widehat{\psi}(l_1, \dots, l_{L+1})(v_0), \quad l_1 \in \mathbf{Z}_N, \end{aligned} \tag{3.25}$$

where l_2, \dots, l_{L+1} are integers with $l_{\ell+1} - l_\ell = \pm 1$ for $1 \leq \ell \leq L$. Note that the number of vectors $v(l_1, \dots, l_{L+1})$'s is strictly greater than 2^N . For a general v_0 , we consider the following linear transformation of $\otimes^L \mathbf{C}^2$ defined by the sum of products of vectors in (3.25),

$$M(v_0) := \sum_{l_1 \in \mathbf{Z}_N} \sum'_{l_\ell \in \mathbf{Z}} v(l_1, \dots, l_{L+1}) h(l_1, \dots, l_{L+1}). \tag{3.26}$$

Hereafter the ‘prime’ summation means $l_{\ell+1} - l_\ell = \pm 1$ for $1 \leq \ell \leq L$, and $l_1 \equiv l_{L+1} \pmod{N}$. The operator $M(v_0)$ is expected to have the rank 2^N , hence to be a non-singular operator. Unfortunately I know of no simple way to prove this condition except checking cases by direct computations (some of which will be given in the appendix). Nevertheless we shall assume the non-singular property of the $\otimes^L \mathbf{C}^2$ -operator (3.26) for the rest of this paper.

We now choose some convenient parameter $\tau_{i,j}, \widehat{\tau}_{i,j}$ in (3.21) to represent the Q -operator. Denote by $\mathbf{Q}_R^0, \mathbf{Q}_L^0$ the operators in (3.10) using the following $\tau_{i,j}$ and $\widehat{\tau}_{i,j}$:

$$\tau_{i,j} = \widehat{\Phi}_{i,j}(v_0), \quad \widehat{\tau}_{i,j} = r_{i,j} \Phi_{l_1, l_2}(v_0) \quad (j - i = \pm 1)$$

where $r_{i,j}$ is defined in (3.23). This means the nonzero $\mathbf{S}_{i,j}^0$ and $\widehat{\mathbf{S}}_{i,j}^0$ in (3.21) are

$$\mathbf{S}_{i,j}^0(v) = \Phi_{i,j}(v) \widehat{\Phi}_{i,j}(v_0), \quad \widehat{\mathbf{S}}_{i,j}^0(v) = \Phi_{l_1, l_2}(v_0) \widehat{\Phi}_{i,j}(v), \quad (j - i = \pm 1). \tag{3.27}$$

In particular, the Q operator in theorem 3.1 can be expressed by

$$Q(v) = \mathbf{Q}_R^0(v) \mathbf{Q}_R^0(v_0)^{-1} = \mathbf{Q}_L^0(v_0)^{-1} \mathbf{Q}_L^0(v).$$

In order to prove theorem 3.1, we need only to verify the relation (2.20), which follows from the equality

$$T^{(N)}(v - 2\eta) \mathbf{Q}_R^0(v_0) = \mathbf{Q}_L^0(v - 2K) \mathbf{Q}_R^0(v), \quad v \in \mathbf{C}, \tag{3.28}$$

with $M_0 = \mathbf{Q}_L^0(v_0)^{-1}$ in (2.20).

Lemma 3.2. *Let $\Phi_{i,j}(v), \widehat{\Phi}_{i',j'}(v)$ be the local vectors in (3.23), and $W^{(N)}(m, m'|l, l')(v - 2\eta)$ the N th fusion weights at $v - 2\eta$ in (3.9). Then for $m, l \in \mathbf{Z}, i, i' \in \mathbf{Z}_N$ such that $\lambda^{(N)}(:=m - l) = N - 1 - 2k$ for some $0 \leq k \leq N - 1$, and $m \equiv i, l \equiv i' \pmod{N}$,*

the equality

$$W^{(N)}(m, m + \lambda|l, l + \mu)(v - 2\eta) = \widehat{\Phi}_{i,i+\lambda}(v - 2K)\Phi_{i',i'+\mu}(v)$$

holds for $\lambda, \mu = \pm 1$.

Proof. For $i, i' \in \mathbf{Z}_N$ and $\lambda, \mu = \pm 1$, by (3.19), one writes the product $\widehat{\Phi}_{i,j}(v - 2K)\Phi_{i',j'}(v)$ in the form

$$\widehat{\Phi}_{i,i+\lambda}(v - 2K)\Phi_{i',i'+\mu}(v) = r_{i,i+\lambda}(2i\eta + \lambda(\eta + v)|2i'\eta + \mu(\eta - v)) = r_{i,i+\lambda}w_{i;i-i'}(\lambda, \mu; v),$$

where the functions $w_{i;n}(\lambda, \mu; v)$ for $i, n \in \mathbf{Z}_N, \lambda, \mu = \pm 1$ are defined by

$$w_{i;n}(\lambda, \mu; v) = \frac{2}{h(K)\Theta(0)}h\left(-K + \left(2i - n + \frac{\lambda + \mu}{2}\right)\eta + \frac{\lambda - \mu}{2}v\right) \times h\left(\left(n + \frac{\lambda - \mu}{2}\right)\eta + \frac{\lambda + \mu}{2}v\right).$$

Comparing the values in (3.9) with $\widehat{\Phi}_{i,j}(v - 2K)\Phi_{i',j'}(v)$ with $j = i + \lambda, j' = i' + \mu$, we find

$$W^{(N)}(l - 1, l|l' - 1, l')(v - 2\eta) = r_{l-1,l}w_{l-1,\lambda^{(N)}}(1, 1; v) = \widehat{\Phi}_{i,i+1}(v - 2K)\Phi_{i',i'+1}(v),$$

$$(l - 1 \equiv i, l' - 1 \equiv i'; \lambda^{(N)} = \mu^{(N)}, \alpha = \beta = 1),$$

$$W^{(N)}(l + 1, l|l' + 1, l')(v - 2\eta) = r_{l+1,l}w_{l+1,\lambda^{(N)}}(-1, -1; v) = \widehat{\Phi}_{i,i-1}(v - 2K)\Phi_{i',i'-1}(v),$$

$$(l + 1 \equiv i, l' + 1 \equiv i'; \lambda^{(N)} = \mu^{(N)}, \alpha = \beta = -1),$$

$$W^{(N)}(l + 1, l|l' - 1, l')(v - 2\eta) = r_{l+1,l}w_{l+1,\lambda^{(N)}}(-1, 1; v) = \widehat{\Phi}_{i,i-1}(v - 2K)\Phi_{i',i'+1}(v),$$

$$(l + 1 \equiv i, l' - 1 \equiv i'; \lambda^{(N)} = \mu^{(N)} + 2, \alpha = -\beta = -1)$$

$$W^{(N)}(l - 1, l|l' + 1, l')(v - 2\eta) = r_{l-1,l}w_{l-1,\lambda^{(N)}}(1, -1; v) = \widehat{\Phi}_{i,i+1}(v - 2K)\Phi_{i',i'-1}(v),$$

$$(l - 1 \equiv i, l' + 1 \equiv i'; \lambda^{(N)} = \mu^{(N)} - 2, \alpha = -\beta = 1).$$

Then follow the results. □

We now show the relation (3.28). Using (3.27), one can express $Q_R^0(v)$ and $Q_L^0(v)$ in terms of product-vectors and covectors in (3.24) and (3.25):

$$Q_R^0(v) = \sum_{i_\ell \in \mathbf{Z}_N, i_1 = i_{L+1}} (\mathbb{S}_{i_1, i_2}^0 \mathbb{S}_{i_2, i_3}^0 \cdots \mathbb{S}_{i_L, i_{L+1}}^0)(v)$$

$$= \sum_{l_1 \in \mathbf{Z}_N} \sum'_{l_\ell \in \mathbf{Z}} \psi(l_1, l_2, \dots, l_{L+1})(v)h(l_1, l_2, \dots, l_{L+1}),$$

$$Q_L^0(v) = \sum_{i_\ell \in \mathbf{Z}_N, i_1 = i_{L+1}} (\widehat{\mathbb{S}}_{i_1, i_2}^0 \widehat{\mathbb{S}}_{i_2, i_3}^0 \cdots \widehat{\mathbb{S}}_{i_L, i_{L+1}}^0)(v)$$

$$= \sum_{m_1 \in \mathbf{Z}_N} \sum'_{m_\ell \in \mathbf{Z}} v(m_1, m_2, \dots, m_{L+1})\widehat{\psi}(m_1, m_2, \dots, m_{L+1})(v), \quad (3.29)$$

with the prime summations as before, i.e., $l_{\ell+1} - l_\ell, m_{\ell+1} - m_\ell = \pm 1$ for $1 \leq \ell \leq L$, and $l_1 \equiv l_{L+1}, m_1 \equiv m_{L+1} \pmod{N}$. By lemma 3.2, the vectors in (3.24) have the following product value:

$$\widehat{\psi}(m_1, m_2, \dots, m_{L+1})(v - 2K)\psi(l_1, l_2, \dots, l_{L+1})(v) = \prod_{\ell=1}^L W^{(N)}(m_\ell, m_{\ell+1}|l_\ell, l_{\ell+1})(v - 2\eta)$$

when $m_1 - l_1 = N - 1 - 2k$ for some $0 \leq k \leq N - 1$. Using (3.29), one finds

$$Q_L^0(v - 2K)Q_R^0(v) = \sum_{l_1 \in \mathbf{Z}_N} \sum'_{m_\ell, l_\ell \in \mathbf{Z}} \left\{ \prod_{\ell=1}^L W^{(N)}(m_\ell, m_{\ell+1}|l_\ell, l_{\ell+1})(v - 2\eta) \right\}$$

$$\times v(m_1, \dots, m_{L+1})h(l_1, \dots, l_{L+1}). \quad (3.30)$$

The vector value of $T^{(N)}(v - 2\eta)Q_R^0(v_0)$ can be obtained by the relation (3.6) for $J = N$:

$$\begin{aligned} & \sum_{l_1 \in \mathbf{Z}_N} \sum'_{l_\ell \in \mathbf{Z}} T^{(N)}(v - 2\eta)\psi(l_1, \dots, l_{L+1})(v_0)h(l_1, \dots, l_{L+1}) \\ &= \sum_{l_1 \in \mathbf{Z}_N} \sum'_{l_\ell, m_\ell \in \mathbf{Z}} \left\{ \prod_{\ell=1}^L W^{(N)}(m_\ell, m_{\ell+1} | l_\ell, l_{\ell+1})(v - 2\eta) \right\} \\ & \quad \times \psi(m_1, \dots, m_{L+1})(v_0)h(l_1, \dots, l_{L+1}), \end{aligned}$$

which is the same as (3.30) by definition of $v(m_1, \dots, m_{L+1})$ in (3.25). This in turn yields the equality (3.28), hence follows theorem 3.1. Note that by (3.25), $M_0^{-1} = Q_L^0(v_0) (= Q_R^0(v_0))$ in (2.20) is equal to the linear transformation $M(v_0)$ in (3.26).

Denote S_R the spatial translation operator of $\otimes^L \mathbf{C}^2$, which takes the k th column to $(k + 1)$ th one for $1 \leq k \leq L$ with the identification $L + 1 = 1$. One has

$$\begin{aligned} S_R \psi(l_1, l_2, \dots, l_{L+1})(v) &= \psi(l_0, l_1, \dots, l_L)(v), \\ S_R v(l_1, l_2, \dots, l_{L+1}) &= v(l_0, l_1, \dots, l_L), \\ \widehat{\psi}(l_1, l_2, \dots, l_{L+1})(v) S_R &= \widehat{\psi}(l_2, l_3, \dots, l_{L+2})(v), \\ h(l_1, l_2, \dots, l_{L+1}) S_R &= h(l_2, l_3, \dots, l_{L+2}), \end{aligned} \tag{3.31}$$

where $l_0 := l_1 - l_{L+1} + l_L$, $l_{L+2} := l_{L+1} + l_2 - l_1$. By (3.29), S_R commutes with $Q_R^0(v)$, $Q_L^0(v)$, and $M(v_0)$, hence

$$[S_R, Q(v)] = 0.$$

4. Concluding remarks

By a similar method of producing Q_{72} -operator in [4], we construct another Q -operator, different from Q_{72} , of the eight-vertex model at the root-of-unity parameter η in (1.1) for the functional-equation study of the eight-vertex model, as an analogy to functional relations in the superintegrable N -state CPM. The Q -operator in this work possesses a structure compatible with the Baxter’s eight-vertex SOS model in [5–7]. By this, using an explicit form of the Q -operator and fusion weights of SOS model in [12, 13], we provide a rigorous mathematical argument to show the functional relations holds under the conjectural non-singular hypothesis (supported by computational evidence in cases) about the operator (3.26). The result could improve our understanding about the newly found ‘root-of-unity’ symmetry of the eight-vertex model [15, 16, 23–27]. In this paper, we study the symmetry of the eight-vertex model by the method of functional relations, which has been the characteristic features in the theory of CPM (see, e.g. [2, 10] and references therein). Together with results previously known in the superintegrable CPM and the root-of-unity six-vertex models [29–33], the conclusion derived from this work has further enhanced the ‘universal role’ of CPM among solvable lattice models in regard to the symmetry of degenerate eigenstates. Although the analysis is done on the case η in (1.1) here, results obtained in [23, 24, 27] for other root-of-unity cases strongly suggest that the analogy can be extended successfully to include all cases in the eight-vertex model with ‘root of unity’ parameter η . Further progress related to the functional relations of the theory will enrich our knowledge about the symmetry of lattice vertex model, and also serve to demonstrate the universal character of CPM.

In this paper, we study problems in the root-of-unity eight-vertex model by far involved only with functional relations and the Q -operator. However the quantitative nature of the

Q -operator in our context depends on the choice of v_0 in (3.27) in performing some explicit calculations. A convenient v_0 has not been found yet, nor the suitable expression of Q -eigenvalues as those for Q_{72} -operator in [23] (2.22)–(2.25). These works remain to be done. Nevertheless, it is suggested by results obtained in the six-vertex model [14, 17, 20, 21, 32, 33] that the understanding of symmetry nature of the eight-vertex model should also be related to the study of degenerated eigenstates using the evaluation function in [15, 16], and the elliptic current operator recently appeared in [27]. The connection with the evaluation function is expected to be found because the better understanding of the symmetry properties depends on it. A process along this line is now under consideration, and further progress would be expected.

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Appendix. Computation of the $M(v_0)$ -operator

In this appendix, we provide some computational evidences about the non-singular property of the operator $M(v_0)$ in (3.26). By (3.31), $M(v_0)$ commutes with the spatial translation operator S_R . We shall decompose $M(v_0)$ as the sum of operators of S_R -eigenspaces so that one can perform the computations on each factor component for cases with small L . For simple notations, we denote

$$V = \bigotimes_{\ell=1}^L \mathbf{C}^2, \quad V^* = \bigotimes_{\ell=1}^L \mathbf{C}^{2*},$$

and write the standard basis elements of V and V^* by

$$|\alpha_1, \dots, \alpha_L\rangle := \otimes_{\ell} |\alpha_{\ell}\rangle \in V, \quad \langle \alpha_1, \dots, \alpha_L | := \otimes_{\ell} \langle \alpha_{\ell} | \in V^*.$$

Let V_k be the S_R -eigenspace of V with the eigenvalue $e^{\frac{2\pi i k}{L}}$; hence, V has the S_R -eigenspace decomposition:

$$V = \sum_{k=0}^{L-1} V_k. \quad (\text{A.1})$$

For $i \in \mathbf{Z}_N$, and $\mu_{\ell} = \pm 1$ for $1 \leq \ell \leq L$ with $\sum_{\ell=1}^L \mu_{\ell} \equiv 0 \pmod{N}$, we denote

$$\begin{aligned} \psi[i; \mu_1, \dots, \mu_L](v) &= \psi(l_1, l_2, \dots, l_{L+1})(v) \in V, \\ \widehat{\psi}[i; \mu_1, \dots, \mu_L](v) &= \widehat{\psi}(l_1, l_2, \dots, l_{L+1})(v) \in V^*, \end{aligned}$$

where $l_1 \equiv i \pmod{N}$, and $l_{\ell+1} - l_{\ell} = \mu_{\ell}$ for $1 \leq \ell \leq L$. As before, \mathbf{C}^N is the space of N -cyclic vectors $w = \sum_{i \in \mathbf{Z}_N} w_i |i\rangle$, and \mathbf{C}^{N*} the space of dual vectors $w^* = \sum_{i \in \mathbf{Z}_N} w_i^* \langle i|$. We define

$$\begin{aligned} \psi[\mu_1, \dots, \mu_L](v) &:= \sum_{i \in \mathbf{Z}_N} \psi[i; \mu_1, \dots, \mu_L](v) \langle i| \in V \otimes \mathbf{C}^{N*}, \\ \widehat{\psi}[\mu_1, \dots, \mu_L](v) &:= \sum_{i \in \mathbf{Z}_N} |i\rangle \widehat{\psi}[i; \mu_1, \dots, \mu_L](v) \in \mathbf{C}^N \otimes V^*. \end{aligned}$$

When $v = v_0$, one has $v(l_1, \dots, l_{L+1}) = \psi(l_1, \dots, l_{L+1})(v_0)$ and $h(l_1, \dots, l_{L+1}) = \widehat{\psi}(l_1, \dots, l_{L+1})(v_0)$ by (3.25), and correspondingly, the vector $v(i; \mu_1, \dots, \mu_L)$, covector $h(i; \mu_1, \dots, \mu_L)$, and $v(\mu_1, \dots, \mu_L) \in V \otimes \mathbf{C}^{N^*}$, $h(\mu_1, \dots, \mu_L) \in \mathbf{C}^N \otimes V^*$. For $\mu_\ell = \pm 1$ ($1 \leq \ell \leq L$) with $\sum_\ell \mu_\ell \equiv 0 \pmod{N}$, we define

$$\begin{aligned} \mathbf{Q}_R^0[\mu_1, \dots, \mu_L] &= \sum_{i \in \mathbf{Z}_N} \psi[i; \mu_1, \dots, \mu_L](v) h[i; \mu_1, \dots, \mu_L] \\ &= \psi[\mu_1, \dots, \mu_L](v) h[\mu_1, \dots, \mu_L], \\ \mathbf{Q}_L^0[\mu_1, \dots, \mu_L] &= \sum_{i \in \mathbf{Z}_N} v[i; \mu_1, \dots, \mu_L] \widehat{\psi}[i; \mu_1, \dots, \mu_L](v) \\ &= v[\mu_1, \dots, \mu_L] \widehat{\psi}[\mu_1, \dots, \mu_L](v), \end{aligned}$$

where each second expression in above is given by evaluating covectors of \mathbf{C}^{N^*} on \mathbf{C}^N -vectors. By (3.31), one finds

$$\begin{aligned} S_R \mathbf{Q}_R^0[\mu_1, \dots, \mu_L](v) S_R^{-1} &= \mathbf{Q}_R^0[\mu_2, \dots, \mu_{L+1}](v), \\ S_R \mathbf{Q}_L^0[\mu_1, \dots, \mu_L](v) S_R^{-1} &= \mathbf{Q}_L^0[\mu_2, \dots, \mu_{L+1}](v), \end{aligned}$$

where $\mu_{L+1} := \mu_1$. Consider the group $\langle S_R \rangle$ generated by S_R , which acts on basis-vectors $|\mu_1, \dots, \mu_L\rangle$ with $\sum_\ell \mu_\ell \equiv 0 \pmod{N}$. Denote by \mathcal{O} the set of $\langle S_R \rangle$ -orbits, i.e. the elements $\mathfrak{o} = \langle S_R \rangle |\mu_1, \dots, \mu_L\rangle$. Define

$$\mathbf{Q}_{R,\mathfrak{o}}^0(v) := \sum_{|\mu_1, \dots, \mu_L\rangle \in \mathfrak{o}} \mathbf{Q}_R^0[\mu_1, \dots, \mu_L](v), \quad \mathbf{Q}_{L,\mathfrak{o}}^0(v) := \sum_{|\mu_1, \dots, \mu_L\rangle \in \mathfrak{o}} \mathbf{Q}_L^0[\mu_1, \dots, \mu_L](v).$$

Then $\mathbf{Q}_{R,\mathfrak{o}}^0, \mathbf{Q}_{L,\mathfrak{o}}^0$ commute with S_R and

$$\mathbf{Q}_R^0(v) = \sum_{\mathfrak{o} \in \mathcal{O}} \mathbf{Q}_{R,\mathfrak{o}}^0(v), \quad \mathbf{Q}_L^0(v) = \sum_{\mathfrak{o} \in \mathcal{O}} \mathbf{Q}_{L,\mathfrak{o}}^0(v).$$

We now express $\mathbf{Q}_{R(v),\mathfrak{o}}^0$ in terms of the vector-decomposition in (A.1). Assume $\mathfrak{o} = \langle S_R \rangle |\mu_1, \dots, \mu_L\rangle$, and let $L_\mathfrak{o}$ be the positive divisor of L such that $\langle S_R^{L_\mathfrak{o}} \rangle$ consists of all elements in $\langle S_R \rangle$ which fix $|\mu_1, \dots, \mu_L\rangle$. Then $\mu_{\ell+L_\mathfrak{o}} = \mu_\ell$ for all ℓ , hence $\sum_{k=0}^{L_\mathfrak{o}-1} \mu_k = \frac{L_\mathfrak{o}}{L} \sum_{k=0}^{L-1} \mu_k \equiv 0 \pmod{N}$, which implies $S_R^{L_\mathfrak{o}} \psi[i; \mu_1, \dots, \mu_L](v) = \psi[i; \mu_1, \dots, \mu_L](v)$. Denote by $\psi[i; \mu_1, \dots, \mu_L]_k(v)$ the V_k -component of $\psi[i; \mu_1, \dots, \mu_L](v)$ in (A.1), and define

$$\psi[\mu_1, \dots, \mu_L]_k(v) = \sum_{i \in \mathbf{Z}_N} \psi[i; \mu_1, \dots, \mu_L]_k(v) |i\rangle \in V \otimes \mathbf{C}^{N^*}.$$

Then $\psi[i; \mu_1, \dots, \mu_L]_k(v) = 0$ except $k =$ a multiple of $\frac{L}{L_\mathfrak{o}}$, therefore

$$\psi[i; \mu_1, \dots, \mu_L](v) = \sum_{k=0}^{L_\mathfrak{o}-1} \psi[i; \mu_1, \dots, \mu_L]_{\frac{kL}{L_\mathfrak{o}}}(v).$$

The same statement holds when replacing ψ by $\widehat{\psi}$. Hence

$$\begin{aligned} \widehat{\psi}[\mu_1, \dots, \mu_L](v) &= \sum_{k=0}^{L_\mathfrak{o}-1} \widehat{\psi}[\mu_1, \dots, \mu_L]_{\frac{kL}{L_\mathfrak{o}}}(v), \\ \psi[\mu_1, \dots, \mu_L](v) &= \sum_{k=0}^{L_\mathfrak{o}-1} \psi[\mu_1, \dots, \mu_L]_{\frac{kL}{L_\mathfrak{o}}}(v), \end{aligned}$$

$$v[\mu_1, \dots, \mu_L] = \sum_{k=0}^{L_o-1} v[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}},$$

$$h[\mu_1, \dots, \mu_L] = \sum_{k=0}^{L_o-1} h[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}.$$

By

$$Q_{R,o}^0(v) = \sum_{m=0}^{L_o-1} S_R^m Q_R^0[\mu_1, \dots, \mu_L](v) S_R^{-m},$$

$$S_R^m Q_R^0[\mu_1, \dots, \mu_L](v) S_R^{-m} = \sum_{k,k'=0}^{L_o-1} e^{\frac{2\pi i m(k-k')}{L_o}} \psi[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}(v) h[\mu_1, \dots, \mu_L]_{\frac{k'L}{L_o}},$$

one finds

$$Q_{R,o}^0(v) = L_o \sum_{k=0}^{L_o-1} \psi[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}(v) h[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}.$$

Similarly, $Q_{L,o}^0(v) = L_o \sum_{k=0}^{L_o-1} v[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}} \widehat{\psi}[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}(v)$. Set $v = v_0$ in the above $Q_R^0(v)$, then follows:

$$M(v_0) = \sum_{o \in \mathcal{O}} M(v_0)_o, \quad M(v_0)_o = L_o \sum_{k=0}^{L_o-1} v[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}} h[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}. \quad (A.2)$$

As the vector and covector in (3.1) are related by $|v\rangle^t = \langle -v|\sigma^x$, one finds

$$r_{i;\mu_1, \dots, \mu_L} \psi[i; \mu_1, \dots, \mu_L](-v)^t = \widehat{\psi}[i; \mu_1, \dots, \mu_L](v)R,$$

where R is the spin-reflection operator in (2.9), and $r_{i;\mu_1, \dots, \mu_L} = \prod_{\ell=1}^L r_{i_\ell, l_{\ell+1}}$ with $l_\ell = i + \sum_{k=1}^{\ell-1} \mu_k$. Hence

$$r_{\mu_1, \dots, \mu_L} \psi[\mu_1, \dots, \mu_L](-v)^t = \widehat{\psi}[\mu_1, \dots, \mu_L](v)R$$

where $r_{\mu_1, \dots, \mu_L} = \sum_{i \in \mathbf{Z}_N} r_{i;\mu_1, \dots, \mu_L} |i\rangle \langle i|$ the non-degenerate diagonal matrix of \mathbf{C}^N . Note that $r_{\mu_1, \dots, \mu_L} = r_{\mu_2, \dots, \mu_{L+1}}$, which depends only the $\langle S_R \rangle$ -orbit o of $|\mu_1, \dots, \mu_L\rangle$. We shall also write $r_o = r_{\mu_1, \dots, \mu_L}$. Then $M(v_0)_o$ is expressed by

$$M(v_0)_o = L_o \sum_{k=0}^{L_o-1} \psi[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}(v_0) r_o \psi[\mu_1, \dots, \mu_L]_{\frac{kL}{L_o}}(-v_0)^t R. \quad (A.3)$$

Since R commutes with S_R , the above summation provides the decomposition of $M(v_0)_o$ on subspaces $V_{\frac{kL}{L_o}}$.

We now use the formulae (A.2) and (A.3) to determine the non-singular property of $M(v_0)$ for $L = 2, 4$, where $\sum_\ell \mu_\ell \equiv 0 \pmod{N}$ is equivalent to $\sum_\ell \mu_\ell = 0$. For convenience, we denote

$$H_i(v) := H((2i + 1)\eta + v), \quad \Theta_i(v) := \Theta((2i + 1)\eta + v), \quad i \in \mathbf{Z}_N.$$

Then $H_i(-v) = -H_{-i-1}(v)$ and $\Theta_i(-v) = \Theta_{-i-1}(v)$. For $L = 2$, \mathcal{O} consists of one $\langle S_R \rangle$ -orbit $o = \{|1, -1\rangle, |-1, 1\rangle\}$ with $L_o = 2$, and the decomposition $V = V_0 + V_1$ of (A.1) is

given by

$$V_0 = \mathbf{C}|1, 1\rangle + \mathbf{C}|-1, -1\rangle + \mathbf{C}(|1, -1\rangle + |-1, 1\rangle), \quad V_1 = \mathbf{C}(|1, -1\rangle - |-1, 1\rangle).$$

With $(\mu_1, \mu_2) = (1, -1)$, one has $\psi[i; 1, -1](v) = \psi(i + 1, i, i + 1)(v)$, equal to $H_i(-v)H_i(v)|1, 1\rangle + H_i(-v)\Theta_i(v)|1, -1\rangle + \Theta_i(-v)H_i(v)|-1, 1\rangle + \Theta_i(-v)\Theta_i(v)|-1, -1\rangle$. Using the equalities (by (3.18))

$$\begin{aligned} H_i(-v)\Theta_i(v) + \Theta_i(-v)H_i(v) &= \frac{2h((2i + 1)\eta)h(v + K)}{\Theta(0)h(K)}, \\ \Theta_i(v)H_i(-v) - H_i(v)\Theta_i(-v) &= \frac{2h((2i + 1)\eta - K)h(v)}{\Theta(0)h(K)}, \end{aligned}$$

we find $\psi[i; 1, -1] = \psi[i; 1, -1]_0 + \psi[i; 1, -1]_1$ with

$$\begin{aligned} \psi[i; 1, -1]_0(v) &= H_i(-v)H_i(v)|1, 1\rangle + \Theta_i(-v)\Theta_i(v)|-1, -1\rangle \\ &\quad + \frac{h((2i + 1)\eta)h(v + K)}{\Theta(0)h(K)}(|1, -1\rangle + |-1, 1\rangle), \\ \psi[i; 1, -1]_1(v) &= \frac{h((2i + 1)\eta - K)h(v)}{\Theta(0)h(K)}(|1, -1\rangle - |-1, 1\rangle). \end{aligned}$$

Hence $\psi[1, -1](v) = \psi[1, -1]_0(v) + \psi[1, -1]_1(v)$ where $\psi[1, -1]_k(v) = \sum_{i \in \mathbf{Z}_N} \psi[i; 1, -1]_k(v)\langle i|$ for $k = 0, 1$. By (A.2) and (A.3),

$$\begin{aligned} M(v_0)R &= 2 \sum_{k=0,1} \psi[1, -1]_k(v_0)r\psi[1, -1]_k(-v_0)^t \\ &= 2 \sum_{k=0,1} (-1)^k \psi[1, -1]_k(v_0)r\psi[1, -1]_k(v_0)^t, \end{aligned}$$

where $r = \sum_{i \in \mathbf{Z}_N} r_{i,i+1}r_{i+1,i}|i\rangle\langle i|$. The non-singular $M(v_0)$ is equivalent to the non-degenerate bilinear form $\psi[1, -1]_k(v_0)r\psi[1, -1]_k(v_0)^t$ of V_k for $k = 0, 1$, which is obvious for $k = 1$. For $k = 0$, $\psi[1, -1]_0(v_0)$ defines three linear independent N -cyclic vectors, then by the nonzero entries of r , follows the non-singular $M(v_0)|_{V_0}$.

We now consider the case $L = 4$, where the decomposition $V = \sum_{k=0}^3 V_k$ in (A.1) is given by

$$\begin{aligned} V_0 &= \mathbf{C}v + \mathbf{C}v' + \mathbf{C}x_0 + \mathbf{C}y_0 + \mathbf{C}w_0 + \mathbf{C}u_0, & V_1 &= \mathbf{C}x_1 + \mathbf{C}y_1 + \mathbf{C}u_1, \\ V_2 &= \mathbf{C}x_2 + \mathbf{C}y_2 + \mathbf{C}w_1 + \mathbf{C}u_2, & V_3 &= \mathbf{C}x_3 + \mathbf{C}y_3 + \mathbf{C}u_3, \end{aligned} \tag{A.4}$$

where

$$\begin{aligned} v &= |1, 1, 1, 1\rangle, & v' &= |-1, -1, -1, -1\rangle, \\ x_n &= \sum_{k=0}^3 i^{nk} S_R^k |1, 1, 1, -1\rangle, & y_n &= \sum_{k=0}^3 i^{nk} S_R^k |1, -1, -1, -1\rangle, \\ w_m &= \sum_{k=0}^1 (-1)^{mk} S_R^k |1, -1, 1, -1\rangle, & u_n &= \sum_{k=0}^3 i^{nk} S_R^k |1, 1, -1, -1\rangle, \end{aligned}$$

for $0 \leq n \leq 3, m = 0, 1$. The transport of the above basis elements, denoted by $v^*, v'^*, x_n^*, y_n^*, w_m^*$ and u_n^* , form a basis of V^* . The set \mathcal{O} in (A.2) consists of two $\langle S_R \rangle$ -orbits:

$$\begin{aligned} \mathfrak{o}_1 &= \{|1, -1, 1, -1\rangle, |-1, 1, -1, 1\rangle\}, & L_{\mathfrak{o}_1} &= 2, \\ \mathfrak{o}_2 &= \{|1, 1, -1, -1\rangle, |-1, 1, 1, -1\rangle, |-1, -1, 1, 1\rangle, |1, -1, -1, 1\rangle\}, & L_{\mathfrak{o}_2} &= 4. \end{aligned}$$

For the orbit \mathfrak{o}_1 represented by $|1, -1, 1, -1\rangle$, one has $\psi[i; 1, -1, 1, -1](v) = \psi(i, i + 1, i, i + 1, i)(v) (= \psi(i, i + 1, i)(v) \otimes \psi(i, i + 1, i)(v))$, which is equal to

$$\begin{aligned}
 &H_i(v)^2 H_i(-v)^2 \mathbf{v} + \Theta_i(v)^2 \Theta_i(-v)^2 \mathbf{v}' + H_i(-v)^2 H_i(v) \Theta_i(v) (|1, 1, 1, -1\rangle + |1, -1, 1, 1\rangle) \\
 &\quad + H_i(-v) H_i(v)^2 \Theta_i(-v) (|-1, 1, 1, 1\rangle + |1, 1, -1, 1\rangle) \\
 &\quad + H_i(-v) \Theta_i(-v) \Theta_i(v)^2 (|1, -1, -1, -1\rangle + |-1, -1, 1, -1\rangle) \\
 &\quad + H_i(v) \Theta_i(-v)^2 \Theta_i(v) (|-1, 1, -1, -1\rangle + |-1, -1, -1, 1\rangle) \\
 &\quad + H_i(-v)^2 \Theta_i(v)^2 |1, -1, 1, -1\rangle + H_i(v)^2 \Theta_i(-v)^2 |-1, 1, -1, 1\rangle \\
 &\quad + H_i(-v) H_i(v) \Theta_i(-v) \Theta_i(v) \mathbf{u}_0.
 \end{aligned}$$

In the basis in (A.4), one finds $\psi[i; 1, -1, 1, -1] = \psi[i; 1, -1, 1, -1]_0 + \psi[i; 1, -1, 1, -1]_2$ where

$$\begin{aligned}
 \psi[i; 1, -1, 1, -1]_0(v) &= H_i(v)^2 H_i(-v)^2 \mathbf{v} + \Theta_i(v)^2 \Theta_i(-v)^2 \mathbf{v}' \\
 &\quad + \frac{H_i(-v)^2 H_i(v) \Theta_i(v) + H_i(-v) H_i(v)^2 \Theta_i(-v)}{2} \mathbf{x}_0 \\
 &\quad + \frac{H_i(-v) \Theta_i(-v) \Theta_i(v)^2 + H_i(v) \Theta_i(-v)^2 \Theta_i(v)}{2} \mathbf{y}_0 \\
 &\quad + \frac{H_i(-v)^2 \Theta_i(v)^2 + H_i(v)^2 \Theta_i(-v)^2}{2} \mathbf{w}_0 + H_i(-v) H_i(v) \Theta_i(-v) \Theta_i(v) \mathbf{u}_0, \\
 \psi[i; 1, -1, 1, -1]_2(v) &= \frac{H_i(-v)^2 H_i(v) \Theta_i(v) - H_i(-v) H_i(v)^2 \Theta_i(-v)}{2} \mathbf{x}_2 \\
 &\quad + \frac{H_i(-v) \Theta_i(-v) \Theta_i(v)^2 - H_i(v) \Theta_i(-v)^2 \Theta_i(v)}{2} \mathbf{y}_2 \\
 &\quad + \frac{H_i(-v)^2 \Theta_i(v)^2 - H_i(v)^2 \Theta_i(-v)^2}{2} \mathbf{w}_1.
 \end{aligned} \tag{A.5}$$

Hence $\psi[1, -1, 1, -1] = \psi[1, -1, 1, -1]_0 + \psi[1, -1, 1, -1]_2$, with

$$\psi[1, -1, 1, -1]_k(v) = \sum_{i \in \mathbb{Z}_N} \psi[i; 1, -1, 1, -1]_k(v) \langle i|, \quad k = 0, 2.$$

By (A.3), one obtains

$$\begin{aligned}
 M(v_0)_{\mathbf{o}_1} R &= 2 \sum_{k=0}^1 \psi[1, -1, 1, -1]_{2k}(v_0) r_{\mathbf{o}_1} \psi[1, -1, 1, -1]_{2k}(-v_0)^t \\
 &= 2 \sum_{k=0}^1 (-1)^k \psi[1, -1, 1, -1]_{2k}(v_0) r_{\mathbf{o}_1} \psi[1, -1, 1, -1]_{2k}(v_0)^t,
 \end{aligned} \tag{A.6}$$

where $r_{\mathbf{o}_1} = \sum_{i \in \mathbb{Z}_N} r_{i,i+1}^2 r_{i+1,i}^2 |i\rangle \langle i|$.

For \mathbf{o}_2 (= the class of $|1, 1, -1, -1\rangle$), $\psi[i; 1, 1, -1, -1](v) = \psi(i, i+1, i+2, i+1, i)(v)$ is expressed by

$$\begin{aligned}
 &H_i(-v) H_{i+1}(-v) H_i(v) H_{i+1}(v) \mathbf{v} + \Theta_i(-v) \Theta_{i+1}(-v) \Theta_i(v) \Theta_{i+1}(v) \mathbf{v}' \\
 &\quad + H_i(-v) H_{i+1}(-v) H_{i+1}(v) \Theta_i(v) |1, 1, 1, -1\rangle \\
 &\quad + H_{i+1}(-v) H_i(v) H_{i+1}(v) \Theta_i(-v) |-1, 1, 1, 1\rangle \\
 &\quad + H_i(-v) H_i(v) H_{i+1}(v) \Theta_{i+1}(-v) |1, -1, 1, 1\rangle \\
 &\quad + H_i(-v) H_{i+1}(-v) H_i(v) \Theta_{i+1}(v) |1, 1, -1, 1\rangle \\
 &\quad + H_i(-v) \Theta_{i+1}(-v) \Theta_i(v) \Theta_{i+1}(v) |1, -1, -1, -1\rangle \\
 &\quad + H_{i+1}(-v) \Theta_i(-v) \Theta_i(v) \Theta_{i+1}(v) |-1, 1, -1, -1\rangle \\
 &\quad + H_{i+1}(v) \Theta_i(-v) \Theta_{i+1}(-v) \Theta_i(v) |-1, -1, 1, -1\rangle
 \end{aligned}$$

$$\begin{aligned}
& + H_i(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_{i+1}(v)|-1, -1, -1, 1\rangle \\
& + H_i(-v)H_{i+1}(v)\Theta_{i+1}(-v)\Theta_i(v)|1, -1, 1, -1\rangle \\
& + H_i(v)H_{i+1}(-v)\Theta_i(-v)\Theta_{i+1}(v)|-1, 1, -1, 1\rangle \\
& + H_i(-v)H_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v)|1, 1, -1, -1\rangle \\
& + H_{i+1}(-v)H_{i+1}(v)\Theta_i(-v)\Theta_i(v)|-1, 1, 1, -1\rangle \\
& + H_i(v)H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v)|-1, -1, 1, 1\rangle \\
& + H_i(-v)H_i(v)\Theta_{i+1}(-v)\Theta_{i+1}(v)|1, -1, -1, 1\rangle.
\end{aligned}$$

In the basis in (A.4), $\psi[i; 1, 1, -1, -1] = \sum_{k=0}^3 \psi[i; 1, 1, -1, -1]_k$ where $\psi[i; 1, 1, -1, -1]_0(v)$

$$\begin{aligned}
& = H_i(-v)H_{i+1}(-v)H_i(v)H_{i+1}(v)v + \Theta_i(-v)\Theta_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v)v' \\
& + (H_i(-v)H_{i+1}(-v)H_{i+1}(v)\Theta_i(v) + H_{i+1}(-v)H_i(v)H_{i+1}(v)\Theta_i(-v) \\
& + H_i(-v)H_i(v)H_{i+1}(v)\Theta_{i+1}(-v) + H_i(-v)H_{i+1}(-v)H_i(v)\Theta_{i+1}(v))\frac{x_0}{4} \\
& + (H_i(-v)\Theta_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) + H_{i+1}(-v)\Theta_i(-v)\Theta_i(v)\Theta_{i+1}(v) \\
& + H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_i(v) + H_i(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_{i+1}(v))\frac{y_0}{4} \\
& + (H_i(-v)H_{i+1}(v)\Theta_{i+1}(-v)\Theta_i(v) + H_i(v)H_{i+1}(-v)\Theta_i(-v)\Theta_{i+1}(v))\frac{w_0}{4} \\
& + (H_i(-v)H_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) + H_{i+1}(-v)H_{i+1}(v)\Theta_i(-v)\Theta_i(v) \\
& + H_i(v)H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v) + H_i(-v)H_i(v)\Theta_{i+1}(-v)\Theta_{i+1}(v))\frac{u_0}{4},
\end{aligned} \tag{A.7}$$

$\psi[i; 1, 1, -1, -1]_1(v)$

$$\begin{aligned}
& = (H_i(-v)H_{i+1}(-v)H_{i+1}(v)\Theta_i(v) - iH_{i+1}(-v)H_i(v)H_{i+1}(v)\Theta_i(-v) \\
& + (-i)^2 H_i(-v)H_i(v)H_{i+1}(v)\Theta_{i+1}(-v) + (-i)^3 H_i(-v)H_{i+1}(-v)H_i(v)\Theta_{i+1}(v))\frac{x_1}{4} \\
& + (H_i(-v)\Theta_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) - iH_{i+1}(-v)\Theta_i(-v)\Theta_i(v)\Theta_{i+1}(v) \\
& + (-i)^2 H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_i(v) + (-i)^3 H_i(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_{i+1}(v))\frac{y_1}{4} \\
& + (H_i(-v)H_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) - iH_{i+1}(-v)H_{i+1}(v)\Theta_i(-v)\Theta_i(v) \\
& + (-i)^2 H_i(v)H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v) + (-i)^3 H_i(-v)H_i(v)\Theta_{i+1}(-v)\Theta_{i+1}(v))\frac{u_1}{4},
\end{aligned} \tag{A.8}$$

$\psi[i; 1, 1, -1, -1]_2(v)$

$$\begin{aligned}
& = (H_i(-v)H_{i+1}(-v)H_{i+1}(v)\Theta_i(v) - H_{i+1}(-v)H_i(v)H_{i+1}(v)\Theta_i(-v) \\
& + H_i(-v)H_i(v)H_{i+1}(v)\Theta_{i+1}(-v) - H_i(-v)H_{i+1}(-v)H_i(v)\Theta_{i+1}(v))\frac{x_2}{4} \\
& + (H_i(-v)\Theta_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) - H_{i+1}(-v)\Theta_i(-v)\Theta_i(v)\Theta_{i+1}(v) \\
& + H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_i(v) - H_i(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_{i+1}(v))\frac{y_2}{4} \\
& + (H_i(-v)H_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) - H_{i+1}(-v)H_{i+1}(v)\Theta_i(-v)\Theta_i(v) \\
& + H_i(v)H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v) - H_i(-v)H_i(v)\Theta_{i+1}(-v)\Theta_{i+1}(v))\frac{u_2}{4} \\
& + (H_i(-v)H_{i+1}(v)\Theta_{i+1}(-v)\Theta_i(v) - H_i(v)H_{i+1}(-v)\Theta_i(-v)\Theta_{i+1}(v))\frac{w_1}{2},
\end{aligned} \tag{A.9}$$

$$\begin{aligned}
 \psi[i; 1, 1, -1, -1]_3(v) &= (H_i(-v)H_{i+1}(-v)H_{i+1}(v)\Theta_i(v) + i(H_{i+1}(-v)H_i(v)H_{i+1}(v)\Theta_i(-v))) \\
 &\quad + i^2(H_i(-v)H_i(v)H_{i+1}(v)\Theta_{i+1}(-v)) + i^3(H_i(-v)H_{i+1}(-v)H_i(v)\Theta_{i+1}(v))) \frac{x_3}{4} \\
 &\quad + (H_i(-v)\Theta_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) + i(H_{i+1}(-v)\Theta_i(-v)\Theta_i(v)\Theta_{i+1}(v))) \\
 &\quad + i^2(H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_i(v)) + i^3(H_i(v)\Theta_i(-v)\Theta_{i+1}(-v)\Theta_{i+1}(v))) \frac{y_3}{4} \\
 &\quad + (H_i(-v)H_{i+1}(-v)\Theta_i(v)\Theta_{i+1}(v) + i(H_{i+1}(-v)H_{i+1}(v)\Theta_i(-v)\Theta_i(v))) \\
 &\quad + i^2(H_i(v)H_{i+1}(v)\Theta_i(-v)\Theta_{i+1}(-v)) + i^3(H_i(-v)H_i(v)\Theta_{i+1}(-v)\Theta_{i+1}(v))) \frac{u_3}{4}.
 \end{aligned} \tag{A.10}$$

Hence $\psi[1, 1, -1, -1](v) = \sum_{k=0}^3 \psi[1, 1, -1, -1]_k(v)$ with

$$\psi[1, 1, -1, -1]_k(v) = \sum_{i \in \mathbb{Z}_N} \psi[i; 1, 1, -1, -1]_k(v) \langle i |, \quad 0 \leq k \leq 3.$$

By (A.3),

$$M(v_0)_{o_2} R = 4 \sum_{k=0}^3 \psi[1, 1, -1, -1]_k(v_0) r_{o_2} \psi[1, 1, -1, -1]_k(-v_0)^t \tag{A.11}$$

where $r_{o_2} = \sum_{i \in \mathbb{Z}_N} r_{i,i+1} r_{i+1,i+2} r_{i+2,i+1} r_{i+1,i} |i\rangle \langle i|$. By (A.6) and (A.11), $M(v_0)R (= M(v_0)_{o_1} R + M(v_0)_{o_2} R)$ is expressed by

$$\begin{aligned}
 2\psi[1, -1, 1, -1]_0(v_0) r_{o_1} \psi[1, -1, 1, -1]_0(v_0)^t &+ 4\psi[1, 1, -1, -1]_0(v_0) r_{o_2} \psi[1, 1, -1, -1]_0(v_0)^t \\
 - 2\psi[1, -1, 1, -1]_2(v_0) r_{o_1} \psi[1, -1, 1, -1]_2(v_0)^t &+ 4\psi[1, 1, -1, -1]_2(v_0) r_{o_2} \psi[1, 1, -1, -1]_2(-v_0)^t \\
 + 4\psi[1, 1, -1, -1]_1(v_0) r_{o_2} \psi[1, 1, -1, -1]_1(-v_0)^t &+ 4\psi[1, 1, -1, -1]_3(v_0) r_{o_2} \psi[1, 1, -1, -1]_3(-v_0)^t,
 \end{aligned} \tag{A.12}$$

which induces the V_k -endomorphism $M(v_0)|_{V_k} R$ for $0 \leq k \leq 3$. The non-singular property of $M(v_0)$ will follow from the non-degeneracy of $M(v_0)|_{V_k} R$ for all k . For $k = 1, 3$, V_k is three dimensional with the basis in (A.4), by which the three N -cyclic vectors to express $\psi[1, 1, -1, -1]_k(v_0)$ in (A.8) and (A.10) are linear independent for a generic v_0 , and the same for $\psi[1, 1, -1, -1]_k(-v_0)$. Therefore, $M(v_0)|_{V_k}$ is a non-singular automorphism of V_k . For $k = 0$, by (A.5) and (A.7), there are six N -cyclic vectors in each of $\psi[1, -1, 1, -1]_0(v_0)$ and $\psi[1, 1, -1, -1]_0(v_0)$ for a generic v_0 . Then by using different quadratic forms of \mathbb{C}^N , they form the V_0 -automorphism $M(v_0)|_{V_0} R$. The nonzero determinant of $M(v_0)|_{V_0} R$ is expected by direct computation for a given N . Also for $k = 2$, one can argue the non-singular property of $M(v_0)|_{V_2}$ in a similar manner. For example in the case $N = 3$, using $H_i(-v) = -H_{-i-1}(v)$ and $\Theta_i(-v) = \Theta_{-i-1}(v)$, one can express (A.5), (A.7) and (A.9) in terms of $H_i(= H_i(v))$, $\Theta_i(= \Theta_i(v))$ for $i = 0, 1, 2$, then obtains

$$\begin{aligned}
 \psi[1, -1, 1, -1]_0(v) &= v \otimes (H_0^2 H_2^2, H_1^4, H_0^2 H_2^2) + v' \otimes (\Theta_0^2 \Theta_2^2, \Theta_1^4, \Theta_0^2 \Theta_2^2) \\
 &\quad + \frac{(H_2^2 H_0 \Theta_0 - H_2 H_0^2 \Theta_2) x_0}{2} \otimes (1, 0, -1) \\
 &\quad + \frac{(-H_2 \Theta_2 \Theta_0^2 + H_0 \Theta_2^2 \Theta_0) y_0}{2} \otimes (1, 0, -1) \\
 &\quad + \frac{w_0}{2} \otimes (H_0^2 \Theta_2^2 + H_2^2 \Theta_0^2, 2H_1^2 \Theta_1^2, H_0^2 \Theta_2^2 + H_2^2 \Theta_0^2) \\
 &\quad - u_0 \otimes (H_0 H_2 \Theta_0 \Theta_2, H_1^2 \Theta_1^2, H_0 H_2 \Theta_0 \Theta_2),
 \end{aligned}$$

$$\begin{aligned}
\psi[1, 1, -1, -1]_0(v) &= (H_0 H_2)v \otimes (H_1^2, H_1^2, H_0 H_2) + (\Theta_0 \Theta_2)v' \otimes (\Theta_1^2, \Theta_1^2, \Theta_0 \Theta_2) \\
&\quad + \frac{(H_2 \Theta_0 - H_0 \Theta_2)H_1^2 x_0}{4} \otimes (1, -1, 0) + \frac{(H_2 \Theta_0 - H_0 \Theta_2)\Theta_1^2 y_0}{4} \otimes (-1, 1, 0) \\
&\quad - \frac{w_0}{4} \otimes (H_1 \Theta_1 (H_2 \Theta_0 + H_0 \Theta_2), H_1 \Theta_1 (H_2 \Theta_0 + H_0 \Theta_2), H_0^2 \Theta_2^2 + H_2^2 \Theta_0^2) \\
&\quad + \frac{(H_0 H_1 \Theta_1 \Theta_2 + H_0 H_2 \Theta_1^2 + H_1 H_2 \Theta_0 \Theta_1 - H_1^2 \Theta_0 \Theta_2)u_0}{4} \otimes (1, 1, 0), \\
\psi[1, -1, 1, -1]_2(v) &= \frac{x_2}{2} \otimes (H_0 H_2^2 \Theta_0 + H_0^2 H_2 \Theta_2, 2H_1^3 \Theta_1, H_0 H_2^2 \Theta_0 + H_0^2 H_2 \Theta_2) \\
&\quad - \frac{y_2}{2} \otimes (H_2 \Theta_0^2 \Theta_2 + H_0 \Theta_0 \Theta_2^2, 2H_1 \Theta_1^3, H_2 \Theta_0^2 \Theta_2 + H_0 \Theta_0 \Theta_2^2) \\
&\quad + \frac{w_1}{2} \otimes (H_2^2 \Theta_0^2 - H_0^2 \Theta_2^2, -2H_1^2 \Theta_1^2, H_0^2 \Theta_2^2 - H_2^2 \Theta_0^2), \\
\psi[1, 1, -1, -1]_2(v) &= \frac{x_2}{4} \otimes (H_1^2 H_2 \Theta_1 + H_0 H_1^2 \Theta_2 - 2H_0 H_1 H_2 \Theta_1, \\
&\quad H_0 H_1 H_2 \Theta_2 + H_0 H_1 H_2 \Theta_1 - H_1^2 H_2 \Theta_0 - H_0 H_1^2 \Theta_2, H_0^2 H_2 \Theta_0 - H_0 H_2^2 \Theta_0) \\
&\quad + \frac{(H_0 \Theta_1^2 \Theta_2 + H_2 \Theta_0 \Theta_1^2 - 2H_1 \Theta_0 \Theta_1 \Theta_2)y_2}{4} \otimes (-1, 1, 0) \\
&\quad + \frac{u_2}{4} \otimes (H_1 H_2 \Theta_0 \Theta_1 + H_1^2 \Theta_0 \Theta_2 + H_0 H_1 \Theta_1 \Theta_2 + H_0 H_2 \Theta_1^2, \\
&\quad H_1 H_2 \Theta_0 \Theta_1 + H_1^2 \Theta_0 \Theta_2 + H_0 H_1 \Theta_1 \Theta_2 + H_0 H_2 \Theta_1^2, 4H_0 H_2 \Theta_0 \Theta_2) \\
&\quad + \frac{w_1}{2} \otimes (-H_1 H_2 \Theta_0 \Theta_1 + H_0 H_1 \Theta_1 \Theta_2, -H_1 H_2 \Theta_0 \Theta_1 \\
&\quad + H_0 H_1 \Theta_1 \Theta_2, -H_0^2 \Theta_2^2 + H_2^2 \Theta_0^2).
\end{aligned}$$

Using (A.12), one finds the non-singular $M(v_0)|_{V_k} R$ when v_0 is generic for $k = 0, 2$. For a general N , the zero-determinant of (A.12) would provide a complicated relation among theta functions $H_i(v)$'s and $\Theta_j(v)$'s, which is unlikely to be true. Unfortunately we cannot provide a rigorous mathematical argument about this statement.

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